Average Rate of change (AROC)

Given a function $y = f(x)$ the average rate of change over the interval (x_1, x_2) is

$$\frac{y_2 - y_1}{x_2 - x_1} = \text{AROC} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example: Find the average rate of change of the function $f(x) = x^2$ over the intervals (0, 1) and (–2, 4).

For (0, 1):

$$\text{AROC} = \frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1 - 0} = 1$$

For (–2, 4):

$$\text{AROC} = \frac{f(4) - f(-2)}{4 - (-2)} = \frac{16 - 4}{6} = \frac{12}{6} = 2$$

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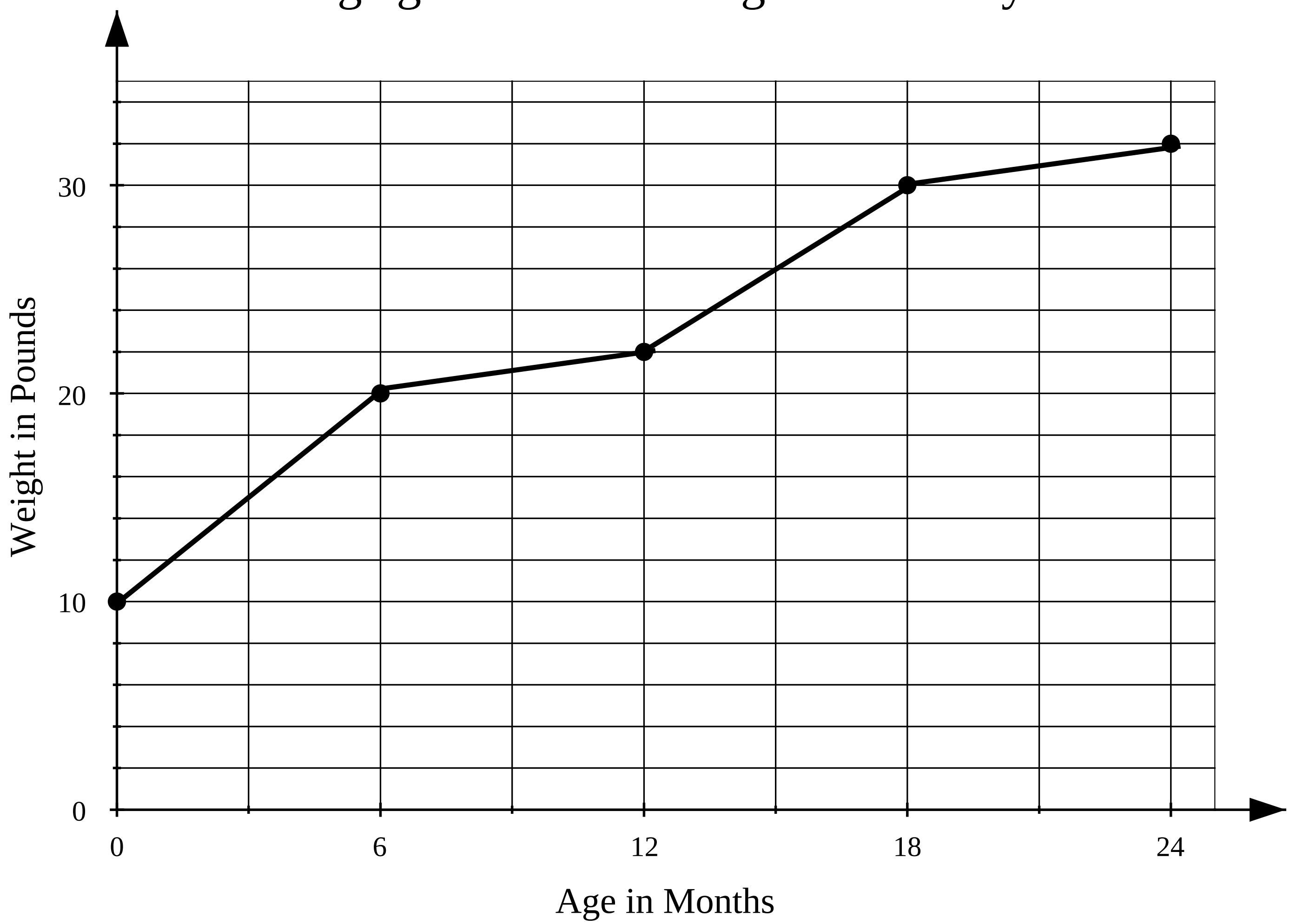
The weight of a racoon, in pounds, is given in the graph below. Use the graph to estimate the average growth rate during the first 6 months, the second half of the first year. Then the average growth rate during the second year.



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First 6 months is the interval [0, 6]

$$\text{AROC} = \frac{f(6) - f(0)}{6 - 0} = \frac{20 - 10}{6} = \frac{5}{3} \text{ lb/month}$$

The next 6 months [6, 12]

$$\text{AROC} = \frac{f(12) - f(6)}{12 - 6} = \frac{22 - 20}{6} = \frac{1}{3} \text{ lb/month}$$

the second year [12, 24]

$$\text{AROC} = \frac{f(24) - f(12)}{24 - 12} = \frac{32 - 22}{12} = \frac{5}{6} \text{ lb/month}$$

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Example: Find the average rate of change of the function $f(x) = 2x^3 - 1$ over the intervals (1, 3) and (2, 5)?

For (1, 3):

$$\text{AROC} = \frac{f(3) - f(1)}{3-1} = \frac{53-1}{2} = 26$$

For (2, 5):

$$\text{AROC} = \frac{f(5) - f(2)}{5-2} = \frac{249-15}{3} = \frac{234}{3} = 78$$

Example: Find the average rate of change of the function $f(x) = 2\sqrt{x}$ over the intervals (1, 4) and (16, 25)?

For (1, 4)

$$\text{AROC} = \frac{f(4) - f(1)}{4-1} = \frac{4-2}{3} = \frac{2}{3}$$

For (16, 25):

$$\text{AROC} = \frac{f(25) - f(16)}{25-16} = \frac{10-8}{9} = \frac{2}{9}$$

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The following table computes the AROC for the function $f(x) = x^2 + x$ beginning at $x = 1$, for small and smaller intervals.

Interval	[1, 1.1]	[1, 1.01]	[1, 1.001]	[1, 1.0001]	[1, 1.00001]
$b - a$	0.1	0.01	0.001	0.0001	0.00001
$f(b) - f(a)$	0.31	0.0301	0.003001	0.00030001	0.0000300001
AROC	3.1	3.01	3.001	3.0001	3.00001

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A moving body's average speed during a time interval is defined as the displacement divided by the elapsed time.

Suppose $s(t) = 16t^2$ is the displacement (say meters) of an object moving on a straight line for a certain time interval (say second), then computing the change in s (denoted Δs) and the time elapsed (denoted Δt,) one can calculate the average velocity in that time interval.

For example the average velocity in the first 2 seconds is

$$\text{AROC} = \frac{\Delta s}{\Delta t} = \frac{16(2^2) - 16(0)}{2 - 0} = 32\,\text{m/s}$$

The average velocity during the 1 second interval between second 1 and second 2 is i.e. (1, 2)

$$\text{AROC} = \frac{16(2^2) - 16(1)}{2 - 1} = 48\text{m}/\text{s}$$

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Consider the general time interval $[t_0, t_0 + \Delta t]$, then

$$\text{AROC} = \frac{16(t_0 + \Delta t)^2 - 16(t_0)^2}{\Delta t} \text{ m/s}$$

Let's compute the average speed at the time $t_0 = 1$ for different values of Δt.

Length of time interval Δt	Average Velocity at $t_0 = 1$
1	48
0.1	33.6
0.01	32.16
0.001	32.016
0.0001	32.0016

The average velocity seems like it gets closer and closer to 32 m/s.

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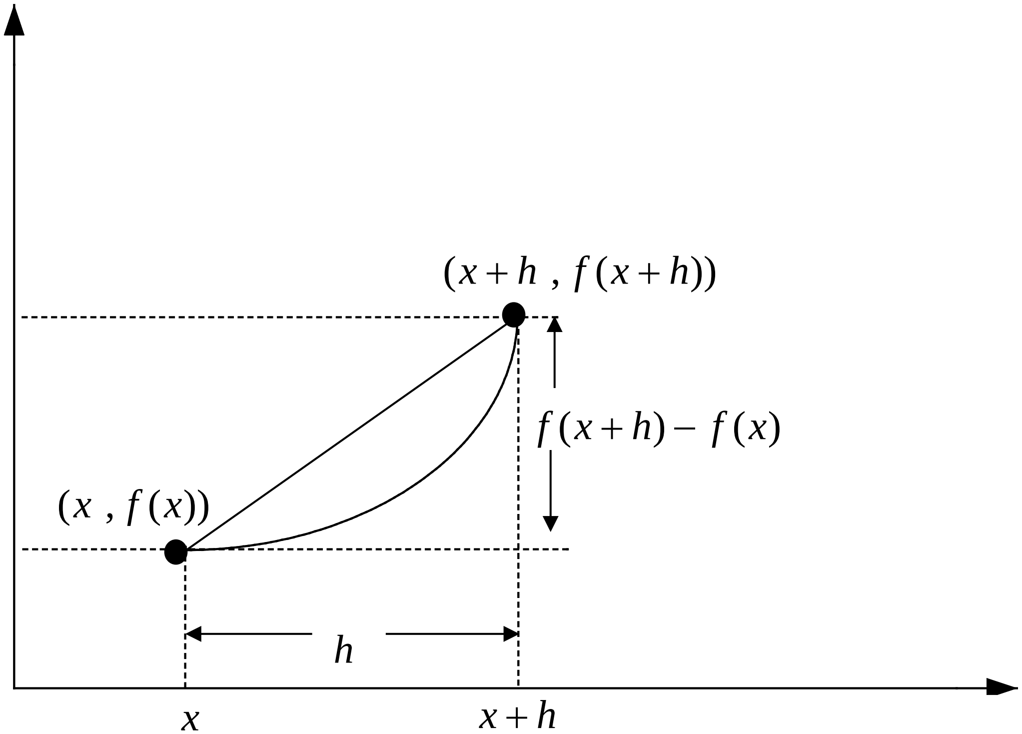
Consider the graph of a function $y = f(x)$ shown below. Suppose P and Q are two points on the graph corresponding to $x_1 = x$, and $x_2 = x + h$.



P is the point $(x\ ,\ f(x))$. Q is the point $(x+h,\ f(x+h))$. The line drawn through these two points is called the **secant line**, and its slope is the **Difference Quotient** (DQ):

$$m_{\text{sec}} = \text{DQ} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

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Find and simplify the DQ for $f(x) = x^2 + x$ on the interval $[1, 1 + h]$

$$\text{D Q} = \frac{f(1+h) - f(1)}{h} = \frac{\left[(1+h)^2 + (1+h)\right] - 2}{h}$$

$$= \frac{\cancel{1} + 2h + h^2 + \cancel{1} + h - \cancel{2}}{h}$$

$$= \frac{3h + h^2}{h} = \frac{\cancel{h}(3+h)}{\cancel{h}} = 3 + h$$

So

$$\text{AROC} = \text{DQ} = (3+h)\ .$$

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Example: The average rate of change of $f(x) = x^2 - x + 2$ over the interval $[-1, -1+h]$ is

$$\text{AROC} = \text{DQ} = \frac{f(-1+h) - f(-1)}{h}$$

Hence

$$\text{AROC} = \text{DQ} = \frac{\left[(-1+h)^2 - (-1+h) + 2\right] - 4}{h}$$

$$= \frac{\cancel{1} - 2h + h^2 + \cancel{1} - h + \cancel{2} - \cancel{4}}{h}$$

$$= \frac{-3h + h^2}{h} = \frac{\cancel{h}(-3+h)}{\cancel{h}} = -3 + h$$

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Example: Find and simplify the difference quotient for $f(x) = x^2 - 2x$

$$\text{DQ} = \frac{f(x+h) - f(x)}{h} = \frac{\left[(x+h)^2 - 2(x+h)\right] - \left[x^2 - 2x\right]}{h}$$

$$= \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h}$$

$$= \frac{2xh + h^2 - 2h}{h} = \frac{h(2x + h - 2)}{h} = 2x + h - 2$$

Memorize the following. You will often use them.

$$(x+h)^2 = x^2 + 2xh + h^2$$

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

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Example: Find and simplify the difference quotient for $f(x) = mx + b$

$$\text{DQ} = \frac{f(x+h) - f(x)}{h} = \frac{[m(x+h)+b] - [mx+b]}{h}$$

$$= \frac{mx + mh + b - mx - b}{h} = \frac{mh}{h} = m$$

This should make sense since *f* is a line with slope *m* everywhere.

Example: Find and simplify the difference quotient for $f(x) = 2x^3$

$$\text{DQ} = \frac{f(x+h) - f(x)}{h} = \frac{[2(x+h)^3] - [2x^3]}{h} = \frac{[2x^3 + 6x^2h + 6xh^2 + 2h^3] - [2x^3]}{h}$$

$$= \frac{[6x^2h + 6xh^2 + 2h^3]}{h}$$

$$= \frac{h[6x^2 + 6xh + 2h^2]}{h} = 6x^2 + 6xh + 2h^2$$

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Example: Find and simplify the difference quotient for $f(x) = x^3 + 3x$

$$\text{DQ} = \frac{f(x+h) - f(x)}{h} = \frac{\left[(x+h)^3 + 3(x+h)\right] - \left[x^3 + 3x\right]}{h}$$

$$= \frac{\left[x^3 + 3x^2h + 3xh^2 + h^3 + 3x + 3h\right] - \left[x^3 + 3x\right]}{h}$$

$$= \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 3x + 3h - x^3 - 3x}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3 + 3h}{h}$$

$$= \frac{h(3x^2 + 3xh + h^2 + 3)}{h} = (3x^2 + 3xh + h^2 + 3)$$

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