Math 8843 - Convergence of Markov Chains
Homework #1

due Monday, October 4

1. In Example 9.4 we studied the spectral gap $\lambda = 1 - \lambda_2$ for the Metropolis chain. Improve the bound to

$$\frac{\max_{a\in\Omega} \frac{\pi(a)}{1/|\Omega|}}{\min_{a\in\Omega} \frac{\pi(a)}{1/|\Omega|}} (1-\lambda_2') \geq 1-\lambda_2 \geq \frac{\min_{a\in\Omega} \frac{\pi(a)}{1/|\Omega|}}{\max_{a\in\Omega} \frac{\pi(a)}{1/|\Omega|}} (1-\lambda_2')\,.$$

Hint: It might be better to consider $c\,f$ or $c+f$ in the denominator of one of the Markov chains.

Remark: If the stationary distributions are the same then it is also possible to compare values of ρ.

2. Show that $\lim_{c\to\infty} Ent((c+f)^2) = 2\,Var_\pi(f)$ and use this to show that $\rho \leq \lambda$, where

$$\rho = \inf_{Ent(f^2)\neq 0} \frac{2\mathcal{E}(f,f)}{Ent(f^2)}\,.$$

3. Determine ρ for the lazy random walk on the boolean cube $\{0,1\}^d$. What can you say about ρ_0? Using these various values, what are the best upper and lower bounds you can show for the mixing time of this Markov chain, in both total variation and L^2 senses?

4. In Theorems 5.1 and 5.2 it was shown that the L^2 distance after t steps can be computed exactly given orthonormal left eigenvectors u_m of the transition matrix P, by the formula

$$\left\|1 - \frac{\mathsf{P}^t(x,\cdot)}{\pi}\right\|_{2,\pi}^2 = \frac{\sum_{m=2}^n \lambda_m^{2t}\,(\overrightarrow{u_m})_x^2}{\pi(x)}$$

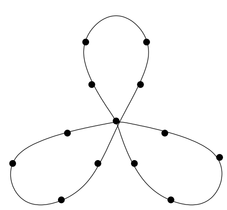
Derive a similar bound in terms of the right eigenvectors of P,

$$\left\|1 - \frac{\mathsf{P}^t(x,\cdot)}{\pi}\right\|_{2,\pi}^2 = \sum_{m=2}^n \lambda_m^{2t}\,(\overrightarrow{w_m})_x^2$$

where the right eigenvectors $(\overrightarrow{w_i})$ are orthonormal with respect to π, that is if $< f,g >_\pi = \sum_{x\in\Omega} \pi(x) f(x) g(x)$ then $< w_i, w_j >_\pi = \delta_{i=j}$.

Remark: Recall that if $f_t = \frac{\mathsf{p}^{(t)}}{\pi}$ then $f_t = \mathsf{P}^t\,f_0$, that is P^t acts on the right on the relative density. The relative density $f_t \xrightarrow{t\to\infty} 1$. In contrast, when we consider $\mathsf{p}^{(t)}$ directly then $\mathsf{p}^{(t)} = \mathsf{p}^{(0)}\,\mathsf{P}^t$ and $\mathsf{p}^{(t)} \xrightarrow{t\to\infty} \pi$. This is why the left eigenvector version divides everything by $\pi(x)$, while the right eigenvector version need not.

5. Consider the simple random walk on the flower F_{mn} with m petals, each a cycle of odd length n.



A vertex is denoted by coordinates (i,j) to denote that it is in petal i (counting counterclockwise from the centerpoint) and vertex j (again counterclockwise), starting at 0 for each. The simple random walk is given by choosing a neighboring vertex uniformly at random and moving there. Recall that the stationary distribution for this walk is proportional to the degrees, in particular, the stationary distribution is $\pi(0) = 1/n$ for the origin (center point) 0, and $\pi(i,j) = 1/mn$ otherwise.

When we learn about walks on symmetric objects we will find that the largest eigenvalue of this is $\cos(\pi/n)$ with multiplicity $(m-1)$, and to each petal $a \in [1 \dots m-1]$ there is associated an orthonormal right eigenvector $\frac{2}{\sqrt{2-3/n}} f_a(i,j)$, where

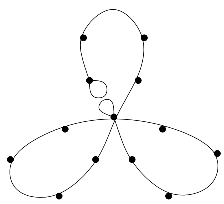
$$f_a(i,j) = \begin{cases} \cos\left(\frac{\pi}{n}(\frac{n}{2} - |j|)\right) & if\ i = a \\ \cos\left(\frac{\pi}{n}(\frac{n}{2} + |j|)\right) & if\ i \neq a \end{cases}$$

Derive matching (up to constant) upper and lower bound on the L^2 mixing time ($\chi^2(\epsilon)$ in Lecture 14) by using the convergence bound proven in the previous problem, as well as methods from earlier in the course. How does this compare to the lower bound on total-variation mixing time?

Hint: Multiplicity of eigenvalues will play a role in the lower bound for L^2 case.

Remark: This is one of the few examples where the total variation and L^2 mixing times are known to be different. The lazy walk on K_n is another such case.

6. Consider the simple random walk on a cut flower $F_{mn} - e$, where some edge e adjacent to the center was removed and replaced by loops.



Use comparison to find upper and lower bounds on the second largest eigenvalue.