Reduction of volume estimation to uniform sampling

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Given a convex body $K \subset \mathbb{R}^n$, we want to estimate the volume.

Remark 22.1. *Generally these volume algorithms can be generalized to integrate log-concave funtions. i.e.*

$$\forall x, y \in K, \lambda \in [0,1] : \lambda \log f(x) + (1-\lambda) \log f(y) \leq \log f(\lambda x - (1-\lambda) y).$$

These include psitive concave functions and standard functions like the Gaussians.

This volume can not be calculated deterministically. For example, consider the following deterministic idea:

1. Put the body in a box $[-c, c]^n$.
2. Check if the points $\{v_1, v_2, \ldots, v_n\}$ are in K, where each $v_i \in \{-\frac{c}{2}, \frac{c}{2}\}$.

This takes 2^n steps, and may be of by a factor of ≥ 2; we can evade any way of choosing points, implying a volume of 0, when in fact the volume is positive. This suggests that any deterministic methodis problematic.

We now try randomization. Consider this simple idea:

1. Put the body in a box as before. $[-c, c]^n$
2. Sample randomly in the box.
3. Let the volume be the fraction of the samples that were in the box, times $(2c)^n$.

This works, unless that fraction is exponentially small. Unfortunately, this often is the case. We now try another method:

1. Put the body in a box. $[-c, c]^n$
2. Create a sequency of boxes B_i, decreasing in volume.
3. Sample from $B_i \cap K$. Notice that this is still convex.
4. Estimate $\frac{Vol(B_{i+1} \cap K)}{Vol(B_i \cap K)} = p_{i+1}$.
5. Let $Vol(K) = \frac{Vol(K \cap B_0}{Vol(K \cap B_1} \cdot \frac{Vol(K \cap B_1)}{Vol(B_2)} \cdot \ldots \cdot \frac{Vol(K \cap B_{k-1})}{Vol(K \cap B_k)} \cdot Vol(B_k)$.

This will be a good estimate for volume if $\frac{1}{c} < p_i \leq c$, for some c.

Note that the easiest way to shrink the side length is by $(1 - \frac{1}{n})$. Then the volume shrinks by $\leq (1 - \frac{1}{n})^n$, which is basically e^{-1}. We'll have $Vol(B_{i+1} \cap K) \geq \frac{Vol(B_i \cap K)}{e}$.

References