Lecture 23 - Introduction to the Ball Walk

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Throughout, a *convex body* $K \subset \mathbb{R}^n$ will mean a closed, bounded convex set. All convex bodies will be given by a "well-guaranteed membership oracle", i.e. we are given that a sphere of radius R centered at the origin contains the body, and a sphere of radius r is contained in the body, with $R, r > 0$, and a black-box oracle, which given a point $x \in \mathbb{R}^n$ says 'yes' if x is in K and 'no' otherwise. Typically $r = 1$ and R is a low degree polynomial in n.

Let $B(x, \delta)$ denote the n dimensional ball of radius δ centered at x. Let $K + \varepsilon$ denote the set of points at distance at most ε from K. Let $Vol_n(A)$ denote the n-dimensional volume of a set A.

Last time it was shown that estimating the volume of a convex body K can be reduced to the problem of sampling uniformly from a convex body. The following algorithms have been proposed for sampling from a convex body.

1. Grid walk [2, 5]: Divide the space $\mathbb{R}^n$ into "very small" cubes. Perform the following random walk on the centers of cubes which intersect $K + \varepsilon$ for some suitably defined ε.
   - Start the walk at the center of the cube containing the origin.
   - Choose a facet of the present cube with probability $1/2n$. If the cube sharing the facet intersects $K + \varepsilon$, move to the cube, otherwise stay in the same cube.
2. Integration [1]: Consider a function which is defined to drop off quickly outside of K. Integrating this function over K will give an estimate on the volume of K. Sampling from such a function can be used to integrate it. It turns out that log-concave functions are a class which can be sampled from, and hence can be used to give estimates on the volume.
3. Ball Walk [6]: From a point $x \in K$ choose a point uniformly at random from $B(x, \delta)$ for some small δ. If the point chosen lies in K, move to it, otherwise stay at x. Intuitively it seems that if we start in a 'corner' then it might take some time to get out of it. The next random walk avoids this problem.
4. Hit-and-Run [7]: From a point $x \in K$ choose a direction uniformly at random and move to a point uniformly along that direction in K.

We will bound the mixing time of the Ball Walk. In order to do this, we first consider the random walk which at each step from the current point x samples uniformly from $K \cap B(x, \delta)$. Of course, we have to ensure that $Vol_n(K \cap B(x, \delta))$ is a large enough fraction of $Vol_n(B(x, \delta))$.

Since the Markov chain is now over a continuous space, we define the transition probability from $x \in K$ to a set A as follows:

$$P(x, A) = \int_{y \in B(x,\delta) \cap K \cap A} \frac{dy}{Vol_n(B(x, \delta) \cap K)}$$

Define the *local conductance* of a point $x \in K$ as

$$\ell(x) = \frac{Vol_n(B(x, \delta) \cap K)}{Vol_n(B(x, \delta))}$$

and let the distribution μ over subsets A of K be

$$\mu(A) = \frac{1}{L} \int_A \ell(x) dx \quad \text{where } L = \int_K \ell(x) dx$$

It can be shown that the samples obtained will be proportional to the local conductance in the limit.

Lemma 23.1. *μ is a stationary distribution for the Markov chain with probabilities given by P.*

Proof. The proof follows Jerrum [3]. If we choose a point x according to the density μ, and make a step according to P and denote the new distribution on sets by μ_1, we have,

$$\begin{aligned}\mu_1(A) &= \int_{y\in A} \mu_1(dy)\\ &= \int_{y\in A}\int_{x\in K} P(x,dy)\mu(dx)\end{aligned}$$

Using the definition for $P(x, dy)$, this is

$$= \int_{y\in A} dy \int_{x\in B(y,\delta)\cap K} \frac{\mu(dx)}{Vol_n(B(x,\delta)\cap K)}$$

Substituting for $\mu(dx)$ we obtain

$$\begin{aligned}&= \frac{1}{L}\int_{y\in A} dy \int_{x\in B(y,\delta)\cap K} \frac{\ell(x)dx}{Vol_n(B(x,\delta)\cap K)}\\ &= \frac{1}{L}\int_{y\in A} dy \int_{x\in B(y,\delta)\cap K} \frac{dx}{Vol_n(B(x,\delta))}\\ &= \frac{1}{L}\int_{y\in A} dy \int_{x\in B(y,\delta)\cap K} \frac{dx}{Vol_n(B(y,\delta))}\\ &= \frac{1}{L}\int_{y\in A} \ell(y)dy\\ &= \mu(A)\end{aligned}$$

□

Theorem 23.2. *[4] There is a constant $c > 0$ so that the mixing time of a Markov chain can be bounded as*

$$\tau(\varepsilon) \le c\int_{\pi_*}^{1/2} \frac{dx}{x(\Phi(x))^2} + \frac{\ln(1/2\varepsilon)}{\Phi^2}$$

where Φ is the average conductance.

For a continuous space chain $\pi_* = 0$ and this does not give a good bound. We will use the following theorem bounding the time to get constant away from stationarity in variation distance instead.

Theorem 23.3.

$$\tau(1/4) \le 15,000\left[\int_{\pi_1}^{1/2} \frac{dx}{x(\Phi(x))^2} + \frac{1}{\Phi}\right]$$

where $\pi_1 = \sup\{t : \forall A \subseteq \Omega \text{ s.t. } \pi(A) = t,\ P(x, A^c) \ge 1/10\ \forall x \in A\}$

We will show that if points are chosen only from $B(x, \delta) \cap K$ then

$$\pi_1 \ge \frac{1}{2}\left(\frac{\delta}{D}\right)^{2n}$$

where $B(0, 1) \subseteq K \subseteq B(0, D/2)$, and that

$$\Phi(x) \ge \min\left\{\frac{1}{288\sqrt{n}}, \frac{\delta}{81\sqrt{n}D}\ln\left(1+\frac{1}{x}\right)\right\}$$

References

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