Lecture 24 - Local conductance and uniform sampling

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Assume the same problem description as in Lecture 23, "Introduction to the Ball Walk".

Much of the material on estimating the volume of a convex body can be found in [1], [2], and [3].

In the last class we began to study the speedy walk where given a poiny $x \in K$ the transition probability to a set A is defined by

$$P(x, A) \quad = \quad \int_{y \in B(x,\delta)\cap K \cap A} \frac{dy}{Vol_n(B(x,\delta)\cap K)}.$$

In the typical case we will set $\delta \leq 1/\sqrt{n}$. It was shown that the stationary distribution μ of this Markov chain is given by

$$\mu(A) = \frac{1}{L}\int_A l(x)dx$$

where $l(x)$ is the local conductance of the point $x \in K$ and is defined by

$$l(x) \quad = \quad \frac{Vol_n(B(x,\delta)\cap K)}{Vol_n(B(x,\delta))}.$$

It is possible for us to modify our sampling procedure in order to ensure that in the limit we will obtain uniform samples from K. Assume that $l(x) \geq c > 0$ for all $x \in K$. Now

(i) Generate a sample according to μ.

(ii) Accept each sample point x with probability $c/l(x)$.

The resulting sampling is uniform since

$$P(\text{sample point } x \in K) \propto l(x) \cdot \frac{c}{l(x)} = c.$$

One way to guarantee that $l(x)$ is sufficiently large is to make some sort of assumption on the boundary of K.

Lemma 24.1. *[1] If for each $x \in K$ there exists a $y \in B(x,\delta)\cap K$ such that $B(y,\delta) \subseteq K$ where $\delta \leq c/\sqrt{n}$, then $.4 \leq l(x) \leq 1$.*

However, the conditions in the lemma are fairly strong so a weaker requirement would be nice. Observe that points fairly far from the boundary of K will be sampled roughly uniformly, and only points near the boundary are heavily non-uniform. This can be exploited by scaling K down by a small factor, only accepting Speedy when it samples in this fairly uniform region, and then scaling the result back out until it entirely covers K.

Theorem 24.2. *[3] Suppose that the Speedy chain is run long enough that the variation distance is at most δ. Then, given a sample $v \in K$ from the Speedy chain check if $\frac{2n}{2n-1} v \in K$; if it is then return $\frac{2n}{2n-1} v$ as a sample from K, otherwise run Speedy again and repeat this procedure. Then if*

$$\delta \leq 1/\sqrt{8n\log(n/\varepsilon)}$$

then the final sample is within 10ε from being uniform.

We now must show that the speedy walk is rapidly mixing. Recall the following from last class for a continuous space chain.

Theorem 24.3.

$$\tau(1/4) \leq 15000 \left[\int_{\pi_1}^{1/2} \frac{dx}{x(\Phi(x))^2} + \frac{1}{\Phi} \right]$$

where $\pi_1 = \sup\{t : \forall A \subseteq \Omega \ s.t. \ \pi(A) = t, P(x, A^C) \geq 1/10 \ \forall x \in A\}$.

This can be used to study $\tau(\varepsilon)$ by a previous inequality from lecture seven.

Lemma 24.4.

$$\tau(\varepsilon) \leq \tau(\delta) \lceil \log_{\frac{1}{2\delta}} (1/2\varepsilon) \rceil.$$

Thus $\tau(\varepsilon) \leq \tau(1/4) \lceil \log_2 (1/2\varepsilon) \rceil$.

We can bound π_1 from below for the speedy walk, so long as the step size δ is sufficiently small. Let D be the diameter of K.

Lemma 24.5. $\pi_1 \leq (1/2)(\delta/D)^{2n}$ *if* $\delta \leq 1$.

Proof. Let $x \in K$ and $S \subseteq K$ be such that $\pi(S) \leq (1/2)(\delta/D)^{2n}$. It suffices to show that $P_x(K \backslash S) \geq 1/10$. Blowing up $K \cap B(x, \delta)$ by a factor of D/δ covers K so that

$$\begin{aligned} l(x) &= \frac{Vol_n(B(x,\delta) \cap K)}{Vol_n(B(x,\delta))} \\ &\geq \left(\frac{\delta}{D}\right)^n \frac{Vol_n K}{Vol_n(B(x,\delta))}. \end{aligned}$$

Using this inequality and the fact that $l(x) \leq 1$ we get

$$\begin{aligned} \pi(S) &= \frac{\int_S l(x)dx}{\int_K l(x)dx} \\ &\geq \frac{\left(\frac{\delta}{D}\right)^n \frac{Vol_n K}{Vol_n B(x,\delta)} Vol_n S}{Vol_n K}. \end{aligned}$$

Thus

$$\begin{aligned} P_x(S) &= \frac{Vol_n(S \cap B(x,\delta))}{Vol_n(K \cap B(x,\delta))} \\ &\leq \frac{Vol_n(S)}{l(x) Vol_n(B(x,\delta))} \\ &\leq \frac{\left(\frac{D}{\delta}\right)^n \pi(S) Vol_n(B(x,\delta))}{\left(\frac{\delta}{D}\right)^n \frac{Vol_n(K)}{Vol_n(B(x,\delta))} Vol_n(B(x,\delta))} \\ &= \pi(S) \left(\frac{D}{\delta}\right)^{2n} Vol_n(B(x,\delta))/Vol_n(K) \\ &\leq 1/2. \end{aligned}$$

□

References

[1] M. Jerrum. *Counting, sampling and integration: algorithms and complexity.* Birkhauser Boston, also see author's website, 2003.

[2] R. Kannan, L. Lovász, and R. Montenegro. Blocking conductance and mixing in random walks. *Preprint*, 2003.

[3] R. Kannan, L. Lovász, and M. Simonovits. Random walks and an $o^*(n^5)$ volume algorithm. *Random Structures and Algorithms*, 1997.