Lecture 25 - Mixing times and conductance

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$$\pi_1 \geq \frac{1}{2}\left(\frac{\delta}{D}\right)^{2n}$$

$$\begin{aligned}
\Phi(x) &\geq \min\{\frac{1}{288\sqrt{n}}, \frac{\delta}{81\sqrt{n}D}\ln(1+\frac{1}{x}\}\\
&= \begin{cases} \frac{1}{288\sqrt{D}} & \text{if } x \lesssim e^{-D/3\delta} \\ \frac{\delta}{81\sqrt{n}D}\ln(1+\frac{1}{x}) & \text{if } x \gtrsim e^{-D/3\delta} \end{cases}
\end{aligned}$$

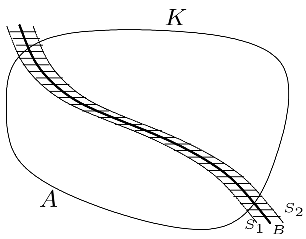
Then

$$\begin{aligned}
\tau(\frac{1}{4}) &\leq 15000\left[288^2 n \int_{\frac{1}{2}(\frac{\delta}{D})^{2n}}^{e^{-D/3\delta}} \frac{dx}{x} + 81^2 n\left(\frac{D}{\delta}\right)^2 \int_{e^{-D/3\delta}}^{\frac{1}{2}} \frac{dx}{x\ln^2 1/x}\right]\\
&= c_1 n \ln x\Big|_{\frac{1}{2}(\frac{\delta}{D})^{2n}}^{e^{-D/3\delta}} + c_2 n\left(\frac{D}{\delta}\right)^2 \frac{1}{\ln 2}\\
&= O\left(n^2 \ln\frac{D}{\delta} + n\left(\frac{D}{\delta}\right)^2\right)\\
&= O(n^3) \qquad \because D = O(\sqrt{n}), \quad \delta = \Omega\left(\frac{1}{\sqrt{n}}\right)
\end{aligned}$$

This is as good as possible, given fixed δ and D. Consider cube of side length D. Diagonal is length $D\sqrt{n}$. Takes $\frac{D\sqrt{n}}{\delta}$ steps to go from one corner to the opposite. about $\left(\frac{1}{2}\frac{D\sqrt{n}}{\delta}\right)^2$ steps to go halfway across cube same as what we showed.

It remains to bound

$$\max_{\substack{\pi(A)\leq x\\ A \text{ closed}}} \frac{Q(A, A^c)}{\pi(A)}.$$



Basic idea: Want to bound $Q(A, A^c) = \frac{1}{2}[Q(A, A^c) + Q(A^c, A)]$. Let

$$\begin{aligned}
S_1 &= \{x \in A | \mathsf{P}_x(A^c) \leq \frac{1}{16}\}\\
S_2 &= \{x \in A^c | \mathsf{P}_x(A) \leq \frac{1}{16}\}\\
B &= K \setminus (S_1 \cup S_2).
\end{aligned}$$

Then

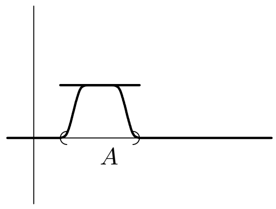
$$Q(A, A^c) \geq \frac{1}{2} \frac{\text{Vol}_n(B)}{16}.$$

It suffices to find how big B is.

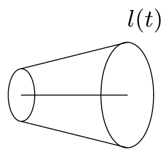
Lemma 25.1. Localization lemma (Lovasz and Simonovitz) *Let g and h be lower semicontinuous (limits of monotone increasing sequence of continuous functions, e.g. indicators of open sets,*

$$\int_{\mathbb{R}^n} g(x)dx > 0 \text{ and } \int_{\mathbb{R}^n} h(x)dx = 0.$$

Then there are two points $a, b \in R^n$ and a linear function $l : [0,1] \to \mathbb{R}_+$ exist such that



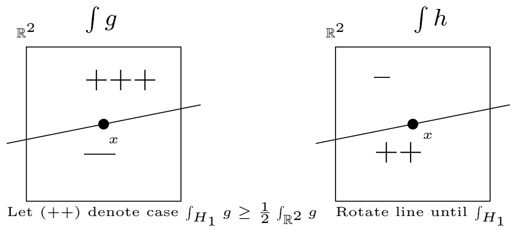
$$\int_0^1 l(t)^{n-1} g((1-t)a + tb)dt > 0 \text{ and } \int_0^1 l(t)^{n-1} h((1-t)a + tb)dt = 0.$$



3d cross sectional area$\propto l(t)^2$. Reduced to needle-like case.

Method of applying: Want to show fact in n-dim. Assume counterexample, writes to integrals. Localization reduces to one dimension. Show impossible in one-dimension.

Sketch of Proof: Suffices to assume g and h are continuous because if $g = \lim_{k\to\infty} g_k$ then $\int g = \liminf g_k$ (Monotone convergence), and likewise for h. To get the result, use a bisection argument (Ham Sandwich)



Either

$$\int_{H1} g \geq \frac{1}{2} \int_{\mathbb{R}^2} g \text{ or } \int_{H_2} g \geq \frac{1}{2} \int_{\mathbb{R}^2} g$$

Let K_1 denote the appropriate half-space, i.e.,

$$\int_{K_1} g(x)dx > 0 \text{ and } \int_{K_1} h(x)dx > 0.$$

Now consider all rational points (countably many) put in a list and bisecting down the list constructing sequence $K_0 \supseteq K_1 \supseteq ....$ Then $\bigcap K_i$ has dimension 0 or 1.