Lecture 26 - Bounding Conductance I

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We continue where we left off last class. Last time we stated the localization lemma, and this time we will be applying it to bound conductance.

Our goal is to show that $\phi(x) \geq \min\{\frac{1}{288\sqrt{n}}, \frac{\delta}{81\sqrt{n}D}\}$.

It suffices to show that if $\pi(A) = x \leq 1/2$ then $Q(A, A^c) \geq \min\{\frac{x}{288\sqrt{n}}, \frac{\delta x}{81\sqrt{n}D}\ln(1+1/x)\} =: A(x)$. i.e. $Q(A, A^c) \geq A(x)$ if $\pi(A) = x$

We write this in terms of integrals over $\mathbb{R}^n$ to apply the Localization Lemma.

Let

$$h(y) = \begin{cases} P_y(A) & if\ y \in A^c, \\ P_y(A^c) & if\ y \in A), \end{cases}$$

Note that this the probability of crossing over.

Therefore, $\int h(y)d\mu(y) = 2Q(A, A^c)$, so it suffices to show that $\int h(y)d\mu(y) \geq 2A$.

We rewrite this in terms of Lebesgue measure, since this is how the localization lemma is stated.

The measure μ, is related to the speedy distribution, so we have $\mu(y) = \frac{l(y)}{\int_K l(y)dy}$, where l denotes the local conductance. Now we want to rewrite the integral, giving us

$$\int h(y)d\mu(y) = \int h(y)\frac{l(y)}{\int_K l(y)dy}dy \geq 2A$$

We can also rewrite the equation $\pi(A) = x$ as

$$\pi(A) = \int \mu(y)\chi_A(y)dy = x$$

To use LL we assume this is false and let S be a set with $\pi(S) = x$ but $Q(A, A^c) < A(x)$.

i.e. $\int h(y)\mu(y)dy < 2A$ and $\int \mu(y)\chi_A(y)dy = x$.

so

$$\int_{\mathbb{R}^n} 2A\mu(y) - h(y)\mu(y)d\mu > 0$$

and

$$\int x\mu(y) - \mu(y)\chi_y(y)dy = 0$$

Note that χ_S is lower semi-continuous if S is closed.

The Localization Lemma says that $\exists a, b \in \mathbb{R}^n$ and a linear function l so $y(t) = (1-t)a + tb$.

So these integrals are equal to:

$$\int_0^1 [2A\mu(t) - h(t)\mu(t)]l(t)^{n-1}dt > 0$$

and

$$\int_0^1 [x\mu(t) - \mu(t)\chi_S(t)]l(t)^{n-1}dt = 0$$

Given a set $T \subseteq [0,1]$ define $\nu(T) = \int_T \mu(t)l(t)^{n-1}dt$ then we have

$$\int_0^1 h(t)d\nu < 2A \int_0^1 \chi_s(t)d\nu = x$$

i.e. Now we have a two dimentional counter example.

We can note that $\mu(t)l(t)^{n-1}$ is a log concave measure (this follows from the fact that local conductance is log concave).

We reduce to the case of a single piece. $J_1 = \{t \in (0,1)|y(t) \in S\}$ and $J_2 = \{t \in (0,1)|y(t) \in S^c\}$

We want to further restrict to point with high flow across: $J_1' = \{t|y(t) \in S$ and $h(t) < 1/16\}$ and $J_2' = \{t|y(t) \in S^c$ and $h(t) < 1/16\}$ $B = [0,1]\backslash(J_1' \cup J_2')$.

We reduce from the general case to the single interval case. We decompose B into maximal intervals $[x_i, y_i]$.

Then $B = \bigcup[x_i, y_i]$

Let $T_i = [0, x_i]$ or $[y_i, 1]$, whichever has a smaller ν measure. Suppose $\nu([x_i, y_i]) \geq C\nu(T_i)$ for some constant C, then $\nu(B) = \sum \nu[x_i, y_i] \geq \sum C\nu(T_i) \geq C(min(\nu(J_1'), \nu(J_2')))$.

There is some interval (y_i, x_{i+1}) which is entirely contained in either J_1' or J_2'.

If we suppose that $[y_i, x_{i+1}] \subseteq J_1'$ then $J_2' \subseteq \cup T_i$ since $\cup T_i \supseteq [0,1]\backslash[x_i, y_{i+1}]$

This implies that $\nu(\cup T_i) \geq min\{\nu(J_1'), \nu(J_2')\}$.