Lecture 27 - Bounding Conductance II

Scribe: Theologos Bountourelis

Let $J_1' = [0, r), B = [r, s], J_2' = (s, 1]$

Corollary 27.1. *Let $a < b < c < d$ and let F be a log-concave function on the interval [a,d]. Let $\nu(a, b) = \int_a^b F(t)dt$. Then*

$$\frac{d-a}{c-b} \geq \frac{\nu(a,b)\nu(c,d)}{\nu(b,c)\nu(a,d)} \ln(1 + \frac{\nu(a,d)^2}{\nu(a,b)\nu(c,d)})$$

Proof. Consider the graph of F (concave).Let $h(t) = \alpha e^{\gamma t}$ be the exponential such that F(b)=h(b),F(c)=h(c). Let

$$\mu(a, b) = \int_a^b h(t)dt$$

Then ,

$$\mu(a, b) \geq \nu(a, b), \mu(c, d) \geq \nu(c, d)$$

but

$$\mu(b, c) \leq \nu(b, c)$$

therefore (it is easier to see if you draw a graph)

$$\frac{\nu(a,b)\nu(c,d)}{\nu(a,d)} \leq \frac{\mu(a,b)\mu(c,d)}{\mu(a,d)} \qquad (1)$$

and

$$\frac{\nu(a,d)}{\nu(b,c)} \leq \frac{\mu(a,d)}{\mu(b,c)}$$

Since $x \ln(1 + \frac{1}{x})$ is increasing , from (1) we get

$$\frac{\nu(a,b)\nu(c,d)}{\nu(a,d)\mu(a,d)} \ln(1 + \frac{\mu(a,d)\nu(a,d)}{\nu(a,b)\nu(c,d)}) \leq \frac{\mu(a,b)\mu(c,d)}{\mu(a,d)^2} \ln(1 + \frac{\mu(a,d)^2}{\mu(a,b)\mu(c,d)}) \qquad (2)$$

If $\nu(a, d) \leq \mu(a, d)$ the left hand side of (2) is greater than

$$\frac{\nu(a,b)\nu(c,d)}{\nu(b,c)\nu(a,d)} \ln(1 + \frac{\nu(a,d)^2}{\nu(a,b)\nu(c,d)})$$

So for the right hand side it suffices to prove

$$\frac{\mu(a,b)\mu(c,d)}{\mu(a,d)^2} \ln(1 + \frac{\mu(a,d)^2}{\mu(a,b)\mu(c,d)}) \leq \frac{d-a}{c-b}$$

(If $\nu(a, d) \geq \mu(a, d)$ then a similar argument works too) or harder

$$\frac{\mu(a,b)\mu(c,d)}{\mu(b,c)\mu(a,d)}\ln(1+\frac{\mu(a,d)^2}{\mu(a,b)\mu(c,d)}) \leq \frac{d-a}{c-b}$$

Recall $\mu(a,b)=\int_a^b \alpha e^{\gamma t}dt = \frac{\alpha}{\gamma}(e^{\gamma b}-e^{\gamma a})$ So we need

$$\frac{(e^{\gamma b}-e^{\gamma a})(e^{\gamma d}-e^{\gamma c})}{(e^{\gamma c}-e^{\gamma b})(e^{\gamma d}-e^{\gamma a})}\ln(1+\frac{(e^{\gamma d}-e^{\gamma a})^2}{(e^{\gamma b}-e^{\gamma a})(e^{\gamma d}-e^{\gamma c})}) \leq \frac{d-a}{c-b}$$

or

$$\frac{(e^{b}-e^{a})(e^{d}-e^{c})}{(e^{c}-e^{b})(e^{d}-e^{a})}\ln(1+\frac{(e^{d}-e^{a})^2}{(e^{b}-e^{a})(e^{d}-e^{c})}) \leq \frac{d-a}{c-b}$$

This can be verified to be true. □

A related result follows

Corollary 27.2. *If F log-concave , l(t) linear and* $\mu(a,b)=\int_a^b F(t)l(t)^{n-1}dt$ *then*

$$\mu(b,c) \geq \frac{1}{4\sqrt{n}}\cdot\frac{|F(b)-F(a)|}{max\{F(b),F(c)\}}\cdot min\{\mu(a,b),\mu(c,d)\}$$

Lets go back to our conductance proof.
By corollary 1 applied to

$$F(t)=g(t)^{n-1}l(t)$$

(g(t) is linear function from localization , l(t) is local conductance) we get

$$\frac{1-0}{s-r} \geq \frac{\nu(J_1')\nu(J_2')}{\nu(B)\nu(I)}\cdot\ln(1+\frac{\nu(I)^2}{\nu(J_1')\nu(J_2')})$$

$$\geq \frac{(\nu(J_1)-\nu(B))(\nu(J_2)-\nu(B))}{\nu(B)\nu(I)}\ln(1+\frac{\mu(I)^2}{\nu(J_1)\nu(J_2)})$$

because of $\nu(B) < 32A\nu(I)$ and $x=\frac{\nu(J_1)}{\nu(I)}=\frac{\int_S l(y)dy}{\int_K l(y)dy}$ we get

$$> \frac{(x-32A)(1-x-32A)}{32A}\ln(1+\frac{1}{x(1-x)})$$

but $A < \frac{x}{288\sqrt{n}} < \frac{x}{288}$ which gives

$$> \frac{x}{81A}\ln(1+\frac{1}{x})$$

but $A \leq \frac{\delta}{81\sqrt{n}D}\cdot x\ln(1+\frac{1}{x})$ and so

$$\geq \frac{D\sqrt{n}}{8}$$

Finally ,

$$s-r < \frac{\delta}{\sqrt{n}D}$$