Lecture 28 - Bounding conductance III

Scribe: Trevis Litherland

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Recall that we had defined the intervals J_1, J_2, and B as follows:

$$\begin{aligned} J_1 &= y([0,1]) \cap S, \\ J_2 &= y([0,1]) \cap S^c, and \\ B &= \{t \in (0,1) : h(y(t)) > \frac{1}{16}\} = [r,s]. \end{aligned}$$

We then showed that $s - r < \delta/\sqrt{n}D$ if there is a counterexample with low conductance. This implies that $|y(s) - y(r)| < (\delta/\sqrt{n}D)|b-a| \leq \delta/\sqrt{n}$.

We now show that the local conductance at s and r actually differ significantly. Choose $u < r$ and $v > s$ such that they are just slightly outside r and s, respectively: $v - u < \delta/\sqrt{n}D$. Now $h(u), h(v) < \frac{1}{16}$ (since u and v are not in B), but we also have that

$$\begin{aligned} h(u) + h(v) &= P_u(S^c) + P_v(S) \\ &= 1 - P_u(S) + P_v(S) \\ &= 1 - (P_u(S) - P_v(S)) \\ &\geq 1 - \max_{A \subset K}\{P_u(A) - P_v(A)\} \\ &= 1 - \|P_u - P_v\|_{TV}. \end{aligned}$$

Let $B(u,\delta)$ and $B(v,\delta)$ be balls of radius δ centered at u and v, respectively. We then have the following result.

Lemma 28.1. *If $\|u - v\|_2 \leq \delta/\sqrt{n}$, then $B(u,\delta) \cap B(v,\delta)$ is large. More specifically, $\|P_u - P_v\| < 1 - \frac{\min\{l(u),l(v)\}}{4\max\{l(u),l(v)\}}$, where l is the local conductance. (Kannan, Lovasz, and Simonovitz, 1997[?]).*

But this implies immediately that $\|P_u - P_v\| > \frac{7}{8}$ and hence that $2\min\{l(u),l(v)\} < \max\{l(u),l(v)\}$.

Now l is continuous, allowing us to extend the above result to r and s by letting u and v go to r and s, respectively. We thus have that $2\min\{l(r),l(s)\} < \max\{l(r),l(s)\}$.

We now use Corollary 2 of the previous lecture, letting $F(y) = l(y)$. Then

$$\begin{aligned} \nu(r,s) &\geq \frac{1}{4\sqrt{n}} \frac{|l(r) - l(s)|}{\max\{l(r),l(s)\}} \min\{\nu(0,r), \nu(s,1)\} \\ &= 1 - \frac{\min\{l(r),l(s)\}}{\max\{l(r),l(s)\}} \\ &\geq \frac{1}{8\sqrt{n}} \min\{\nu(0,r), \nu(s,1)\} \\ \Rightarrow \nu(B) &\geq \frac{1}{8\sqrt{n}} \min\{\nu(J_1'), \nu(J_2')\}, \end{aligned}$$

where as before $J_1' = [0,r)$, and $J_2' = (s,1]$.

Now we clearly have that

$$\begin{aligned}
\nu(J_1') &\geq \nu(J_1) - \nu(B), and \\
\nu(J_2') &\geq \nu(J_2) - \nu(B) \\
\Rightarrow \nu(B) &\geq \frac{1}{8\sqrt{n}}[\min\{\nu(J_1), \nu(J_2)\} - \nu(B)] \\
\Rightarrow \nu(B) &\geq \frac{1}{9\sqrt{n}} \min\{\nu(J_1), \nu(J_2)\} \\
&= \frac{1}{9\sqrt{n}} x\nu(I).
\end{aligned}$$

Recall that our 1-D counterexample had $\nu(B) < 32A\nu(I)$, with $A = \min\{\frac{x}{288\sqrt{n}}, \frac{\delta}{81\sqrt{n}D} ln(1+1/x)\} < \frac{x}{9\sqrt{(n)}}\nu(I)$, giving us a contradiction.

We summarize the results of the these lectures on bounding conductance as follows:

1) We assumed that our convex body could be transformed into the case with $B(0,1) \subseteq K \subseteq B(0,R)$ for $R = O(\sqrt{n})$. (See KLS, '97 in RSA[?]).

2) We next showed that it suffices to sample uniformly from a convex body.

3) We then stated that uniform samples could be obtained by sampling from local conductance (i.e. using the Speedy Walk). (See KLS[?], or Jerrum if body is essentially round[?]).

4) We showed that the Speedy Walk mixes fast with $\delta = c/\sqrt{n}$ by showing high conductance. This involved the following:

a) We assumed a counterexample $S \subseteq K$, with S of measure x, and then wrote $Q(S, S^c) < A$, $\int_S l(y)\,dy = x\int_K l(y)\,dy$ as integrals over $\mathbb{R}^n$, and finally used The Localization Lemma to show that there is a one-dimensional counterexample satisfying $\nu(B) < 32A\nu(I)$.

b) In this one-dimensional case, we then split the interval into (possibly) many intervals of high/low flow to the complement.

c) We showed that if this counterexample is the case that B had a single interval, then it would be impossible in the case of many intervals. Hence, it sufficed to consider the single interval case.

d) We showed that if the single interval case holds, then there is no counterexample. This required that we demonstrate that the endpoints of B have very different local conductance (i.e. the endpoints were close but had very different single step distributions). Corollary 2 then gave us that $\nu(B)$ must be fairly large relative to the set of size x.

References