Lecture 6 - The mixing time of simple random walk on a cycle

Friday, August 27

To date the only Markov chain for which we know much about the mixing time is the walk on the uniform two-point space. Today we use Theorem 2 of the previous lecture to find the mixing time of a non-trivial Markov chain.

Consider the simple random walk on the cycle C_ℓ (equivalently, on $\mathbb{Z}_\ell$). This has transition matrix

$$\mathsf{P}(i,j) = \begin{cases} \frac{1}{2} & \text{if } j \equiv i \pm 1 \mod \ell \\ 0 & \text{otherwise} \end{cases}$$

Fix some point on the cycle and label it as 0, then number the remaining points in the clockwise direction up to $\ell - 1$.

If ℓ is even then this walk is periodic and the Markov chain does not converge, so let us restrict our attention to the case when ℓ is odd. In this case the eigenvalue / eigenvector pairs are given by

eigenvalue	eigenvector(s)
1	*constant*
$\forall 1 \le j \le \frac{\ell-1}{2} : \cos\left(\frac{2\pi j}{\ell}\right)$	$\cos\left(\frac{2\pi\, j\, k}{\ell}\right),\ \sin\left(\frac{2\pi\, j\, k}{\ell}\right)$

where $0 \le k \le \ell - 1$ is the position around the cycle. This is easily verified to be a valid eigenbasis.

Now, in order to apply Theorem 2 we require orthonormal eigenvectors of the matrix given by $S_{ij} = \sqrt{\frac{\pi(i)}{\pi(j)}}\,\mathsf{P}_{ij}$. But, π is uniform and so this reduces to $S = \mathsf{P}$ and the eigenvectors given above for P will suffice. In general, if π is uniform then $S = \mathsf{P}$.

It remains to make these orthonormal. Clearly the constant eigenvector should be $\overrightarrow{u_1} = 1/\sqrt{\ell}$. Also, eigenvectors with different eigenvalues are always orthogonal.

Now, consider the eigenvector $\overrightarrow{v} = \cos\left(\frac{2\pi\, j\, k}{\ell}\right)$. This has norm

$$\begin{aligned} \overrightarrow{v} \cdot \overrightarrow{v} = & \sum_{k=0}^{\ell-1} \cos^2\left(\frac{2\pi\, j\, k}{\ell}\right) & = \sum_{k=0}^{\ell-1} \frac{1 + \cos\left(\frac{4\pi\, j\, k}{\ell}\right)}{2} \\ = & \ \frac{\ell}{2} + \frac{1}{2}\sum_{k=0}^{\ell-1} \cos\left(\frac{4\pi\, j\, k}{\ell}\right) & = \frac{\ell}{2} + \frac{1}{2}\,\Re\left(\sum_{k=0}^{\ell-1} \exp\left(i\,\frac{4\pi\, j\, k}{\ell}\right)\right) \end{aligned}$$

where the second equality used the identity $\cos^2 x = \frac{1+\cos 2x}{2}$ and $\Re(x)$ denotes the real part of x. If $4j$ and ℓ are relatively prime then this is a sum over the ℓ-th roots of unity $\xi_1, \xi_2 \ldots, \xi_\ell$. But this is just zero if $\ell > 1$, because

$$x^\ell - 1 = (x - \xi_1)(x - \xi_2)\cdots(x - \xi_\ell) = x^\ell - \left(\sum_{i=1}^{\ell} \xi_i\right) x^{\ell-1} + \cdots + (-1)^\ell \prod_{i=1}^{\ell} \xi_i$$

and the coefficient of $x^{\ell-1}$ is zero if $\ell - 1 > 0$. If $4j$ and ℓ are not relatively prime then the sum is $(4j, \ell)$ times the sum of the $\frac{\ell}{(4j,\ell)}$ roots of unity, and everything still adds to zero. In short, $\overrightarrow{v} \cdot \overrightarrow{v} = \ell/2$.

A similar argument holds for the sinusoidal eigenvectors, using the identity $\sin^2 x = \frac{1-\cos 2x}{2}$.

Finally, eigenvectors with the same eigenvalue satisfy $\sum_{k=0}^{\ell-1} \sin\left(\frac{2\pi\, j\, k}{\ell}\right) \cos\left(\frac{2\pi\, j\, k}{\ell}\right) = \frac{1}{2}\sum_{k=0}^{\ell-1} \sin\left(\frac{4\pi\, j\, k}{\ell}\right) = \frac{1}{2}\Im\left(\sum_{k=0}^{\ell-1} \exp\left(\frac{4\pi\, j\, k}{\ell}\right)\right) = 0$, where $\Im$ denotes the imaginary part of the sum of roots of unity.

The orthonormal eigenbasis is then the basis given earlier, normalized by a factor of $\sqrt{2/\ell}$.

Finally the L^2 distance can be determined.

$$\begin{aligned}
\left\| 1 - \frac{\mathsf{P}^t(x,\cdot)}{\pi(x)} \right\|_{2,\pi}^2 &= \frac{\sum_{m=1}^{n} \lambda_m^{2t}\, (\overrightarrow{u_m})_x^2}{\pi(x)} - 1 \\
&= \frac{1}{1/\ell} \sum_{j=1}^{(\ell-1)/2} \cos^2\left(\frac{2\pi\, j}{\ell}\right) \frac{2}{\ell} \left(\cos^2\left(\frac{2\pi\, j\, k}{\ell}\right) + \sin^2\left(\frac{2\pi\, j\, k}{\ell}\right)\right) \\
&= 2 \sum_{j=1}^{\lfloor \frac{\ell-1}{4} \rfloor} \cos^{2t}\left(\frac{2\pi\, j}{\ell}\right) + 2 \sum_{j=1+\lfloor \frac{\ell-1}{4} \rfloor}^{(\ell-1)/2} \cos^{2t}\left(\frac{2\pi\, j}{\ell}\right) \\
&= \boxed{2 \sum_{j=1}^{(\ell-1)/2} \cos^{2t}\left(\frac{\pi\, j}{\ell}\right) = \left\| 1 - \frac{\mathsf{P}^t(x,\cdot)}{\pi(x)} \right\|_{2,\pi}^2}
\end{aligned}$$

The final equality applied the identity $\cos(\pi(1-x)) = -\cos(\pi\, x)$ to the second sum.

This is our first non-trivial example where we could determine a distance exactly. However, in its current form this is not particularly useful since we have no notion of how large the sum is. We now simplify this via a procedure suggested in Diaconis' book [1].

Observe that $\cos x \leq e^{-x^2/2}$ when $x \in [0, \pi/2]$. This follows by letting $h(x) = \log(e^{-x^2/2} \cos x)$, then $h'(x) = x - \tan x \leq 0$ for $x \in [0, \pi/2]$ and so $h(x) \leq h(0) = 0$. It follows that

$$\begin{aligned}
\sum_{j=1}^{(\ell-1)/2} \cos^{2t}\left(\frac{\pi\, j}{\ell}\right) &\leq \sum_{j=1}^{(\ell-1)/2} e^{-t\pi^2 j^2/\ell^2} \\
&\leq e^{-\pi^2\, t/\ell^2} \sum_{j=1}^{\infty} e^{-\pi^2 (j^2-1) t/\ell^2} \\
&\leq e^{-\pi^2\, t/\ell^2} \sum_{j=0}^{\infty} e^{-3\pi^2 j\, t/\ell^2} \\
&= \frac{e^{-\pi^2\, t/\ell^2}}{1 - e^{-3\pi^2 t/\ell^2}}
\end{aligned}$$

where the first inequality applied $\cos x \leq e^{-x^2/2}$, the second factored out a term and extended the sum, the third used the inequality $j^2 - 1 \geq 3(j-1)$ for $j \in \mathbb{Z}_{>0}$ (the case $j = 1$ is trivial, and $j \geq 2$ is simple algebra), and the final equality is because this was a geometric series.

For a final simplification observe that if $t \geq \frac{\ell^2}{3\pi^2} \log 2 \approx \frac{\ell^2}{40}$ then $e^{-3\pi^2 t/\ell^2} \leq 1/2$. Therefore

$$\|\mathsf{P}^t(x,\cdot) - \pi\|_{TV} \leq \frac{1}{2} \left\| 1 - \frac{\mathsf{P}^t(x,\cdot)}{\pi} \right\|_{2,\pi} \leq e^{-\pi^2\, t/2\ell^2} \text{ if } t \geq \ell^2/40$$

To finish this off, solve $e^{-\pi^2\, t/\ell^2} \leq \epsilon$ to find that

$$\boxed{\tau(\epsilon) \leq \frac{2\ell^2}{\pi^2} \log \frac{1}{\epsilon} \quad \text{if } \epsilon \leq e^{-\pi^2/80} \approx 0.88}$$

In the next lecture we show a matching lower bound, showing that this upper bound is essentially correct.

References

[1] P. Diaconis. *Group Representations in Probability and Statistics*, volume 11 of *Lecture Notes - Monograph Series*. Institute of Mathematical Statistics, 1988.