

New Cheeger bounds
on eigenvalues of Markov chains

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(Discrete) Markov Chains

- Random walk on a graph.
- Stochastic matrix \mathbf{P} : row sums 1.
- Irreducible ($\forall i, j \exists t : \mathbf{P}_{ij}^t > 0$)
 \Rightarrow unique stationary distribution $\pi \mathbf{P} = \pi$.
- Time-reversible ($\pi_i \mathbf{P}_{ij} = \pi_j \mathbf{P}_{ji}$)
 $\Rightarrow 1 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n \geq -1$

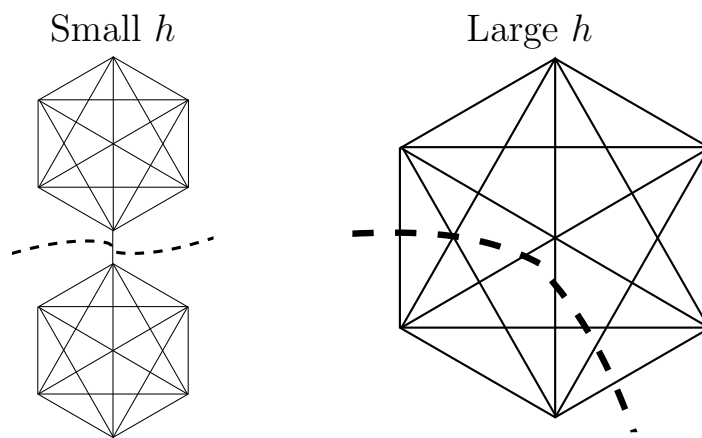
Mixing

- Irreducible + Aperiodic $\Rightarrow \mathbf{p}^{(t)} \xrightarrow{t \rightarrow \infty} \pi$
- Variation distance $\|\sigma - \pi\|_{TV} = \frac{1}{2} \sum_{v \in V} |\sigma(v) - \pi(v)|$.
- $\Delta(t) = \max_{\sigma_0} \|\sigma_0 \mathbf{P}^t - \pi\|_{TV} = \max_{x \in V} \|\mathbf{P}^t(x, \cdot) - \pi\|_{TV}$
- $\frac{1}{2} \lambda_{max}^t \leq \Delta(t) \leq \frac{1}{2} \lambda_{max}^t \sqrt{\frac{1 - \pi_*}{\pi_*}}$ with $\lambda_{max} = \max\{\lambda_2, |\lambda_n|\}$
- Math 8843 notes

Cheeger's Inequality

- (Ergodic) flow : $Q(A, C) = \sum_{i \in A, j \in C} \pi_i P_{ij}$.
- Cheeger constant :

$$h = \min_{A \subset V} \frac{Q(A, A^c)}{\min\{\pi(A), \pi(A^c)\}}$$



Cheeger Inequality

(LAWLER-SOKAL '88, JERRUM-SINCLAIR '89)

$$1 - 2h \leq \lambda_2 \leq 1 - h^2/2$$

(CHUNG, HOUDRÉ-TETALI, ...)

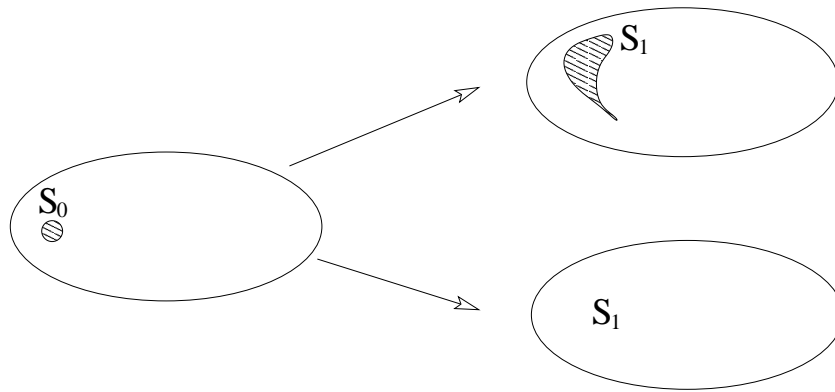
$$\lambda_2 \leq \sqrt{1 - h^2}$$

Evolving Sets

Morris and Peres (STOC 2003) “Evolving sets and mixing”

- $S_0 = \{x\}$
- Choose $u \in [0, 1]$ u.a.r.
- Let $A = S_n$. Then

$$S_{n+1} = A_u = \{v \in V : Q(A, v) \geq u \cdot \pi(v)\}$$



Key Properties

- $P^t(x, y) = \frac{\pi(y)}{\pi(x)} P(y \in S_t \mid S_0 = \{x\})$
- $E\pi(S_t) = \pi(S_0)$
- If \mathcal{M} is lazy ($\forall v \in V : P(v, v) \geq 1/2$) then

$$Q(A, A^c) = \int_0^{1/2} (\pi(A_u) - \pi(A)) du = \int_{1/2}^1 (\pi(A) - \pi(A_u)) du$$

Mixing

Chi-Square Mixing

$$\begin{aligned}
 \Delta(t) &= \max_{x \in V} \|\mathbf{P}^t(x, \cdot) - \pi\|_{TV} \leq \max_{x \in V} \frac{1}{2} \|\mathbf{P}^t(x, \cdot) - \pi\|_{\chi^2(\pi)}^{1/2} \\
 &\leq \max_{x \in V} \frac{1}{2\pi(x)} \mathbf{E} \sqrt{\min\{\pi(S_t), 1 - \pi(S_t)\}} \\
 &\leq \frac{\psi^t}{2\sqrt{\pi_*}} \quad \text{with} \quad \psi = \max_{\substack{A \subset V, \\ \pi(A) \leq 1/2}} \int_0^1 \sqrt{\frac{\pi(A_u)}{\pi(A)}} du.
 \end{aligned}$$

Cheeger Corollary

$$\lambda_2 \leq \lambda_{max} \leq \psi$$

If \mathcal{M} is lazy

$$\psi \leq \frac{\sqrt{1+2h} + \sqrt{1-2h}}{2} \leq 1 - \frac{h^2}{2}$$

Proof: Apply $\frac{1}{2} \lambda_{max}^t \leq \Delta(t)$. Then use Jensen's inequality.

Montenegro "Evolving set bounds on various mixing quantities"

Mixing Theorem

$$\begin{aligned}
 \|\mathbf{P}^t(x, \cdot) - \pi\|_{TV} &\leq \frac{1}{\pi(x)} \mathbf{E} \pi(S_t) (1 - \pi(S_t)) \\
 Ent_{\pi} \left(\frac{\mathbf{P}^t(x, \cdot)}{\pi} \right) &\leq \frac{1}{\pi(x)} \mathbf{E} \pi(S_t) \log \frac{1}{\pi(S_t)} \\
 \|\mathbf{P}^t(x, \cdot) - \pi\|_{\chi^2(\pi)}^{1/2} &\leq \frac{1}{\pi(x)} \mathbf{E} \sqrt{\pi(S_t) (1 - \pi(S_t))}
 \end{aligned}$$

Extending Cheeger

Montenegro “Evolving set and Cheeger bounds on eigenvalues of Markov chains”

Theorem Given $f : (0, 1) \rightarrow \mathbb{R}_{>0}$ let $f(0) := f(1) := 0$. Then

$$\lambda_{max} \leq \mathcal{C}_f = \max_{A \subset V} \int_0^1 \frac{f(\pi(A_u))}{f(\pi(A))} du$$

If $\forall x \in (0, 1/2) : f(x) \leq f(1-x)$ then consider only $\pi(A) \leq 1/2$.

Proof: Let $M = \max_{\substack{A \subset V, \\ \pi(A) \neq 0, 1}} \frac{\sqrt{\min\{\pi(A), 1 - \pi(A)\}}}{f(\pi(A))} \leq 1/\sqrt{2}$. Then

$$\begin{aligned} \|\mathbf{P}^t(x, \cdot) - \pi\|_{TV} &\leq \frac{1}{2\pi(x)} \mathbb{E} \sqrt{\min\{\pi(S_t), 1 - \pi(S_t)\}} \\ &\leq \frac{M}{\pi(x)} \mathbb{E} f(\pi(S_t)) \\ &\leq \frac{M f(\pi(x))}{\pi(x)} \mathcal{C}_f^t. \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \lambda_{max}^t \leq \Delta(t) &\leq \left(\max_{x \in V} \frac{M f(\pi(x))}{\pi(x)} \right) \mathcal{C}_f^t \\ &\implies \lambda_{max} \leq \mathcal{C}_f \end{aligned}$$

Cheeger Corollary

$$\lambda_2 \leq \lambda_{max} \leq \psi$$

Proof: Let $f(x) = \sqrt{x}$. Then use Jensen’s inequality.

Separating λ_2 from λ_n

- Given general (non-lazy) P . Consider $P' = \frac{1}{2}(I + P)$.

$$\lambda'_i = \frac{1}{2}\lambda_i + \frac{1}{2} \quad \Rightarrow \lambda_2 \leq -1 + 2\mathcal{C}'_f$$

- Given P with $\forall v \in V : P(v, v) \geq \gamma$. Consider $P'' = \frac{1}{1-\gamma}P + \left(1 - \frac{1}{1-\gamma}\right)I$.

$$\lambda''_i = \frac{1}{1-\gamma}\lambda_i - \frac{\gamma}{1-\gamma} \quad \Rightarrow \lambda_n \geq -(1-\gamma)\mathcal{C}''_f + \gamma$$

Example: Morris-Peres bound for lazy chain is $\lambda_2 \leq 1 - h^2/2$.

Consider general \mathcal{M} .

Rescale. Then $Q'(A, A^c) = \frac{1}{2}Q(A, A^c)$ so $h' = h/2$.

$$\Rightarrow \lambda_2 \leq -1 + 2\mathcal{C}'_f \leq 1 - h^2/4$$

Extending Cheeger

Corollary Given $f : (0, 1) \rightarrow \mathbb{R}_{>0}$ concave let $f(0) := f(1) := 0$. Then

$$\lambda_2 \leq -1 + 2 \max_{ACV} \frac{f(\pi(A) + Q(A, A^c)) + f(\pi(A) - Q(A, A^c))}{2 f(\pi(A))}$$

$$\lambda_n \geq - \max_{ACV} \frac{\wp f\left(\pi(A) + \frac{\Psi(A)}{\wp}\right) + (1 - \wp) f\left(\pi(A) - \frac{\Psi(A)}{1 - \wp}\right)}{f(\pi(A))}$$

where $\Psi(A) = \frac{1}{2} \int_0^1 |\pi(A) - \pi(A_u)| du$ and $\wp = \sup\{y : \pi(A_y) \geq \pi(A)\}$.

If $\forall x \in [0, 1/2] : f(x) \leq f(1 - x)$ then it suffices to consider $\pi(A) \leq 1/2$.

Proof: Apply Jensen and rescale for λ_2 . Apply Jensen directly for λ_n .

Remarks

- $Q(A, A^c)$ is ergodic flow from A to A^c .
 $\Psi(A)$ is smallest flow from A to a set of size $\pi(A^c)$.
- For lazy chains $\wp = 1/2$ and $\Psi(A) = Q(A, A^c)$.
- $Q(A, A^c) = 0$ iff disconnected.
 $\Psi(A) = 0$ iff A is a bipartition.

Examples

- **Periodic walk on uniform two-point space**
 Corollary is always sharp ($-1 \leq \lambda_n = \lambda_2 \leq -1$).
- **Complete graph or weighted two-point space**
 \mathcal{C}_f form is always sharp if $f(x)$ symmetric.

Cheeger-like Inequalities

Corollary

$$1 - \tilde{h} \leq \lambda_2 \leq -1 + 2\sqrt{1 - \tilde{h}^2/4} \leq 1 - \frac{\tilde{h}^2}{4}$$

$$\lambda_n \geq -\sqrt{1 - \tilde{\phi}^2} \geq -1 + \frac{\tilde{\phi}^2}{2}$$

where

$$\tilde{h} = \min_{A \subset V} \frac{Q(A, A^c)}{\pi(A)\pi(A^c)} \quad \text{and} \quad \tilde{\phi} = \min_{A \subset V} \frac{\Psi(A)}{\pi(A)\pi(A^c)}.$$

Proof: Let $f(x) = \sqrt{x(1-x)}$ and simplify.

Corollary

$$\lambda_2 \leq 1 - \mathbf{g}_1^2$$

$$\lambda_n \geq -1 + 2\mathbf{g}_2^2$$

where

$$\mathbf{g}_1 = \min_{\pi(A) \leq 1/2} \frac{Q(A, A^c)}{\pi(A)\sqrt{1 + \log(1/2\pi(A))}} \quad \text{and} \quad \mathbf{g}_2 = \min_{\pi(A) \leq 1/2} \frac{\Psi(A)}{\pi(A)\sqrt{1 + \log(1/2\pi(A))}}.$$

Proof: Let $f(x) = x(1 + \log(1/2x))$ and simplify.

Corollary

$$\lambda_2 \leq \max_{A \subset V} \cos(2\pi Q(A, A^c))$$

$$\lambda_n \geq -\max_{A \subset V} \cos(2\pi \Psi(A)) \quad \text{if } \varphi = 1/2 \text{ for all } A \subset V$$

Proof: Let $f(x) = \sin(\pi x)$ and simplify.

How to choose $f(x)$?

Lemma If f and f'' are concave then

$$\lambda_2 \leq 1 - \min_{A \subset V} \frac{-f''(\pi(A))}{f(\pi(A))} \mathbf{Q}(A, A^c)^2.$$

Proof: Check that $f(x+y) + f(x-y) \leq 2f(x) + f''(x)y^2$ if f and f'' are concave.

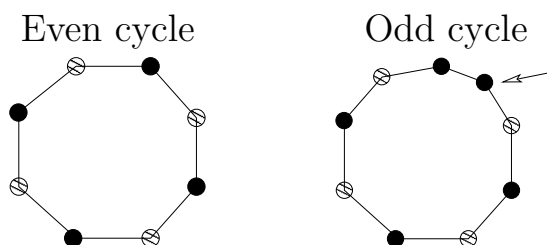
Strategy Given $\mathbf{Q}(A, A^c) \geq C g(\pi(A))$, let $c > 0$ and solve

$$g(x)^2 f''(x) + c f(x) = 0$$

If $f : (0, 1) \rightarrow \mathbb{R}_{>0}$ and f and f'' concave then

$$\lambda_2 \leq 1 - c C^2$$

Simple walk on cycle C_n



- n even: $\mathbf{Q}(A, A^c) \geq 1/n$, $\Psi = 0$

Solution to $f''(x) + \pi^2 f(x) = 0$ is $f(x) = \sin(\pi x)$.

$$\lambda_2 \leq \cos(2\pi/n) \quad \text{and} \quad \lambda_n \geq -1$$

- n odd: $\mathbf{Q}(A, A^c) \geq 1/n$, $\Psi(A) \geq 1/2n$ with $\varphi = 1/2$

$$\lambda_2 \leq \cos(2\pi/n) \quad \text{and} \quad \lambda_n \geq -\cos(\pi/n)$$

All are correct values!

Vertex Expansion

Alon, Houdré-Tetali, Stoyanov, ...

λ_2 for a graph in terms of

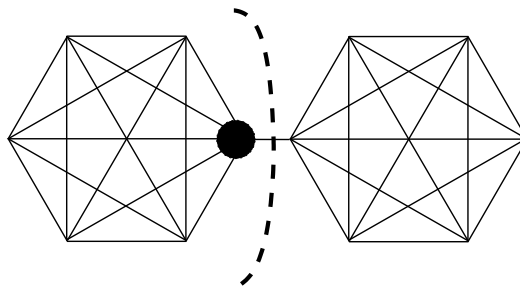
- $\partial_{in}(A) = \{x \in A : P(x, A^c) > 0\}, \quad \partial_{out}(A) = \{x \in A^c : P(x, A) > 0\}$
- $h_{in} = \min_{\substack{A \subset V \\ \pi(A) \leq 1/2}} \frac{\pi(\partial_{in}(A))}{\pi(A)}, \quad h_{out} = \min_{\substack{A \subset V \\ \pi(A) \leq 1/2}} \frac{\pi(\partial_{out}(A))}{\pi(A)}$
- $P_0 = \min_{x, y \in V} \{P(x, y) : P(x, y) > 0\}.$

Max-degree walk on graph

$$1 - \lambda_2 \geq \max \left\{ \frac{P_0}{2} \left(1 - \sqrt{1 - h_{in}}\right)^2, \frac{P_0}{4} \left(\sqrt{1 + h_{out}} - 1\right)^2 \right\}$$

$$\geq \max \left\{ \frac{P_0 h_{in}^2}{8}, \frac{P_0 h_{out}^2}{16} \left(1 - \frac{h_{out}}{4}\right)^2 \right\}$$

Example Max degree walk on Barbell $K_n - K_n$.



$$h = \frac{1}{n^2} \Rightarrow 1 - \lambda_2 \geq \frac{1}{2n^4}$$

$$h_{in} = 1/n, \quad P_0 = 1/n \Rightarrow 1 - \lambda_2 \geq \frac{1}{8n^3}.$$

If $h = P_0 h_{in}$ then

$$1 - \lambda_2 \geq \frac{h^2}{8P_0}$$

Much bigger than Cheeger if P_0 small.

Easy Inequality Prover

Lemma If $f : [0, 1] \rightarrow \mathbb{R}$ concave, two non-increasing functions $g, g' : [0, 1] \rightarrow [0, 1]$ satisfy $\int_0^1 g(u) du = \int_0^1 g'(u) du$ and $\forall t \in [0, 1] : \int_0^t g(u) du \geq \int_0^t g'(u) du$, then

$$\int_0^1 f \circ g(u) du \leq \int_0^1 f \circ g'(u) du.$$

Strategy

- Given initial conditions (e.g. \mathcal{M} is lazy and $\pi(A)$ and $Q(A, A^c)$ known) let $g(u) = \pi(A_u)$.
- $\exists g(u)$ satisfying conditions and minimizes $\int_0^t g(u) du$?
- If so then this maximizes \mathcal{C}_f for any choice of $f(x)$ concave!

Example Chain \mathcal{M} lazy and $\pi(A)$ and $Q(A, A^c)$ known.

- Average value of $\pi(A_u)$ for $u \leq 1/2$ is $\pi(A) + \frac{Q(A, A^c)}{1/2}$.
- $\pi(A_u)$ decreasing $\Rightarrow \forall t \leq 1/2 : \int_0^t \pi(A_u) du$ dominates average.
- Then $\pi(A_u) = \pi(A) + 2Q(A, A^c)$ for all $u \in [0, 1/2]$ will maximize \mathcal{C}_f .
- Likewise, $\pi(A_u) = \pi(A) - 2Q(A, A^c)$ for all $u \in (1/2, 1]$ maximizes \mathcal{C}_f .

$\Rightarrow \mathcal{C}_f$ maximized by

$$\pi(A_u) = \begin{cases} \pi(A) - 2Q(A, A^c) & \text{if } u > 1/2 \\ \pi(A) + 2Q(A, A^c) & \text{if } u \leq 1/2 \end{cases}$$

for every concave f .

Easy Inequalities

Lazy and h known

Worst case $\pi(A_u)$ just shown. Then

$$\begin{aligned} \lambda_2 \leq \mathcal{C}_{\sqrt{x}} &\leq \max_{\pi(A) \leq 1/2} \frac{1}{2} \sqrt{\frac{\pi(A) + 2\mathbf{Q}(A, A^c)}{\pi(A)}} + \frac{1}{2} \sqrt{\frac{\pi(A) - 2\mathbf{Q}(A, A^c)}{\pi(A)}} \\ &= \frac{\sqrt{1 + 2h} + \sqrt{1 - 2h}}{2} \end{aligned}$$

ϕ known

- Fix $\pi(A)$. Then $\Psi(A) = \phi(A) \pi(A)$.
- Consider intervals $u \in [0, \phi]$ and $u \in [\phi, 1]$.

$\implies \mathcal{C}_f$ maximized by

$$\pi(A_u) = \begin{cases} \pi(A) - \frac{\Psi(A)}{1-\phi} & \text{if } u > \phi \\ \pi(A) + \frac{\Psi(A)}{\phi} & \text{if } u \leq \phi \end{cases}$$

\implies Cheeger bound on λ_n .

Given h_{in} , h_{out} and h

Assume \mathcal{M} is lazy. Worst case is

$$\pi(A_u) = \begin{cases} \pi(A) - h_{in} \pi(A) & \text{if } u \in [1 - \mathbf{P}_0, 1] \\ \pi(A) - 2\pi(A) \frac{h - \mathbf{P}_0 h_{in}}{1 - 2\mathbf{P}_0} & \text{if } u \in [1/2, 1 - \mathbf{P}_0] \\ \pi(A) + 2\pi(A) \frac{h - \mathbf{P}_0 h_{out}}{1 - 2\mathbf{P}_0} & \text{if } u \in [\mathbf{P}_0, 1/2] \\ \pi(A) + h_{out} \pi(A) & \text{if } u \in [0, \mathbf{P}_0] \end{cases}$$

Bound λ_2 . Rescale for general case.

$$\begin{aligned} \implies \lambda_2 \leq & -1 + \mathbf{P}_0 \left(\sqrt{1 - h_{in}} + \sqrt{1 + h_{out}} \right) \\ & + (1 - \mathbf{P}_0) \left(\sqrt{1 - \frac{h - \mathbf{P}_0 h_{in}}{1 - \mathbf{P}_0}} + \sqrt{1 + \frac{h - \mathbf{P}_0 h_{out}}{1 - \mathbf{P}_0}} \right) \end{aligned}$$

Other Inequalities

Given h_{out}

Let $h = P_0 h_{out}$ and $h_{in} = h = P_0 h_{out}$.

$$\begin{aligned} \implies \lambda_2 &\leq \sqrt{1 - P_0 h_{out}} + P_0 \left(\sqrt{1 + h_{out}} - 1 \right) \\ &\leq 1 - \frac{P_0 h_{out}^2}{8} \left(1 + P_0 - \frac{h_{out}}{2} \right) \end{aligned}$$

Given \tilde{h}_{in}

Proceed as before, but consider $\tilde{h}_{in}(A)$ and then $\mathcal{C}_{\sqrt{x(1-x)}}$.

$$\begin{aligned} \lambda_2 &\leq \sqrt{1 - \left(\frac{\tilde{h}_{in} P_0}{2} \right)^2} + P_0 \left(\sqrt{1 - \left(\frac{\tilde{h}_{in}}{2} \right)^2} - 1 \right) \\ &\leq 1 - \frac{\tilde{h}_{in}^2 P_0}{8} (1 + P_0) \end{aligned}$$

Consider $\mathcal{C}_{\sqrt{x(1-x)}}$ on previous page.

$$\begin{aligned} \lambda_2 &\leq -1 + P_0 \sqrt{(1 - h_{in}) \left(1 + h_{in} \frac{\pi_*}{1 - \pi_*} \right)} + P_0 \sqrt{(1 + h_{out}) \left(1 - h_{out} \frac{\pi_*}{1 - \pi_*} \right)} \\ &\quad + (1 - P_0) \sqrt{\left(1 - \frac{h - P_0 h_{in}}{1 - P_0} \right) \left(1 + \frac{h - P_0 h_{in}}{1 - P_0} \frac{\pi_*}{1 - \pi_*} \right)} \\ &\quad + (1 - P_0) \sqrt{\left(1 + \frac{h - P_0 h_{out}}{1 - P_0} \right) \left(1 - \frac{h - P_0 h_{out}}{1 - P_0} \frac{\pi_*}{1 - \pi_*} \right)} \end{aligned}$$

Periodic walk on uniform two-point space.

Proof of mixing theorem

Theorem (Montenegro) Every (non-lazy, non-reversible) Markov chain satisfies

$$\|\mathbf{P}^t(x, \cdot) - \pi\|_{TV} \leq \frac{1}{\pi(x)} \mathbf{E}\pi(S_t)(1 - \pi(S_t))$$

Proof: Let $\mathbf{K}^n(S_0, S_n)$ be probability of S_n given S_0 .
Let $f(x) = (1 - x)^+ = (1 - x) \delta_{x \leq 1}$. Then

$$\begin{aligned} \|\mathbf{p}^{(n)} - \pi\|_{TV} &= \sum_{y \in V} f\left(\frac{\mathbf{p}^{(n)}(y)}{\pi(y)}\right) \pi(y) && \text{set up problem} \\ &= \sum_{y \in V} f\left(\frac{\mathbf{P}_{\{x\}}(y \in S_n)}{\pi(x)}\right) \pi(y) && \text{introduce evolving sets} \\ &= \sum_{y \in V} f\left(\sum_{S_n \subset V} \frac{\mathbf{1}_{y \in S_n}}{\pi(S_n)} \frac{\pi(S_n) \mathbf{K}^n(\{x\}, S_n)}{\pi(x)}\right) \pi(y) && \text{rewrite with indicator} \\ &\leq \sum_{y \in V} \sum_{S_n \subset V} f\left(\frac{\mathbf{1}_{y \in S_n}}{\pi(S_n)}\right) \frac{\pi(S_n) \mathbf{K}^n(\{x\}, S_n)}{\pi(x)} \pi(y) && \text{pull out sum with Jensen} \\ &= \sum_{S_n \subset V} \left[\sum_{y \in V} f\left(\frac{\mathbf{1}_{y \in S_n}}{\pi(S_n)}\right) \pi(y) \right] \frac{\pi(S_n) \mathbf{K}^n(\{x\}, S_n)}{\pi(x)} && \text{change order of summation} \\ &= \sum_{S_n \subset V} (1 - \pi(S_n)) \frac{\pi(S_n) \mathbf{K}^n(\{x\}, S_n)}{\pi(x)} && \text{evaluate sum} \end{aligned}$$

Remarks

Theorem (Zhang)

$$-\sqrt{1 - \xi^2} \leq \lambda_n \leq -1 + 4\xi$$

where

$$\xi = \min_{S_1, S_2 \subset V} \frac{Q(S_1, S_1) + Q(S_2, S_2) + Q(S_1 \cup S_2, V \setminus (S_1 \cup S_2))}{\pi(S_1 \cup S_2)}.$$

Conjecture If $|\lambda_n| > \lambda_2$ then $\phi < h$.

Conjecture The second smallest eigenvalue of the Laplacian of a graph is

$$\begin{aligned} \lambda_2 &\geq 2 \left[1 - \sqrt{1 - (\tilde{i}_\infty^+/2)^2} + \Delta \left(1 - \sqrt{1 - \left(\frac{\tilde{i}_\infty^+}{2\Delta}\right)^2} \right) \right] \\ \lambda_2 &\geq 2 \left[1 - \sqrt{1 + i_\infty^-} + \Delta \left(1 - \sqrt{1 - i_\infty^-/\Delta} \right) \right] \end{aligned}$$