

Compounding and Discounting

Introduction

Many introductory finance courses cover the concepts of compounding and discounting. Sometimes these topics are referred to as “Time Value of Money”, and they play a central role in finance, a field where there is a heavy emphasis placed on cash flows obtained at different points in time. Often, the general idea is summarized with the proverb: “A dollar today is worth more than the same dollar in the future”. It is useful to ask why this should be true. One answer is that **inflation** can erode the value of a dollar. If the prices of goods increase, it will cost more money in the future to buy the same goods.

But it is also true that even in the absence of inflation, a dollar today is worth more. Consider the example of buying a CD player today, or buying an identical CD player for the same amount of cash one year later. If you wait, you would forgo one year’s use of the CD player. With a few exceptions, people have a preference for immediate consumption.

Another important reason for the preference for today’s dollars is the possibility of the productive use of the funds. Every investment in production usually requires some capital. The farmer needs to buy land, seeds, equipment, etc. His hope is to realize more money than the initial invested funds. If you lend out money today, you forgo the possibility of earning additional funds yourself, but you expect to be compensated for it in terms of receiving a greater amount in the future. This would be true even if you expect inflation to be zero for the period your money is lent out.

You could describe the money lent out, and the repayment of the loan with a pair of cash flows, or you could summarize that information with a single number indicating the equivalent rate of return. Since these cash flows are related to a loan, the rate of return is also called the interest rate. If the interest rate is agreed on ahead of time, this would be the stated or **nominal interest rate**. This rate combines the compensation for the foregone consumption and investment opportunities along with the compensation for future inflation. The first component is called the **real rate of interest** and it is combined with the rate of inflation to make up the nominal interest rate.¹

A further complication in finance is the possibility that future cash flows are uncertain. This is not considered for now, but will be covered in future sections. If the cash flows are uncertain, we would like to be compensated with an even higher rate of return. If the cash flows we are considering are risk free, the nominal rate of return is then called the risk free rate of return. If the cash flows are uncertain, then the additional rate for compensating for risk is referred to as the **risk premium** and is added to the risk premium.

¹ The mathematics of combining these rates are discussed in the next document called “Return Calculations”. Usually the nominal rate is the only observed rate, and if we know the inflation rate, we can back out the real rate of return. It is generally a constant positive number, indicating that even in the absence of inflation there is a positive time value of money.

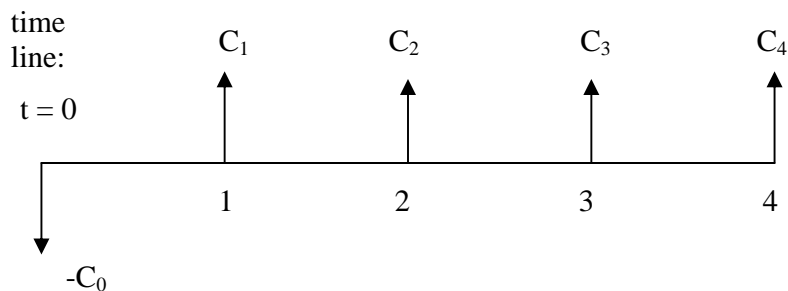
If we have a cash flow today, we might be interested in the equivalent value of the funds sometime in the future. If we consider the interest rate for the opportunity cost of the funds, we would call the equivalent value the **future value** at time t . **Compounding** means that interest is paid not only on the **principal** (the original investment), but also on accumulated and unpaid previous interests. The term **discounting** is related to finding the equivalent **present values** ($t = 0$) of future cash flows in today's dollars. Besides the need to do these calculations for a number of financial assets such as bonds, mortgages, and other interest rate type securities, understating this material is important because many of the concepts are similar to return calculations for other financial assets.

As you go through these documents, look for these connections because they will help you see finance as one unified topic rather than a set of disjointed tasks, each requiring some unique formula to obtain a solution. You will be able to see the solution rooted in intuition!

Some of the mathematical notations used here are reviewed in the appendix to the notes.

Compounding and Future Value:

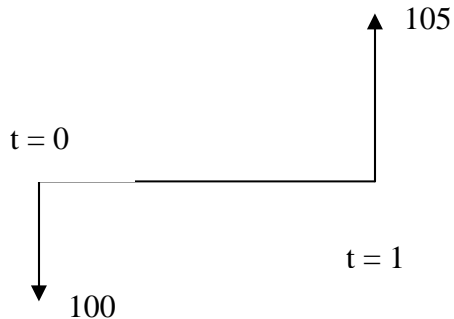
As mentioned above, there is a heavy emphasis in finance on the cash flows which occur at various points in time. It is very useful to represent these cash flows with a time line or cash flow diagram:



Negative cash flow C_0 at $t = 0$
positive cash flow C_t at $t = 1, 2, 3,$ and 4 .

The above diagram indicates that at time $t = 0$, which generally represents today or the present, a negative cash flow of C dollars occurred. The negative or outflow of dollars is reinforced with either the negative sign or the downward pointing arrow (or both!). At time $t = 1, 2, 3,$ and 4 , four positive or inflow cash flows occurred. They were all of different magnitude, indicated by the different subscripts on each C .

Suppose that you invest \$100 in a savings account at your local bank, and it pays you 5% interest per year. If you remove your money after one year:



The cash flow at $t = 1$ is the return of your original capital, or the principal, plus the interest owed for the single year:

$$C_1 = 100 + 100(.05) = 100 + 5 = \$105 \quad (\text{principal} + \text{interest})$$

In general, if C_0 represents the $t = 0$ cash flow, after one single period at rate r :

$$C_1 = C_0 + C_0(r) \quad \text{or} \quad C_1 = C_0(1 + r)$$

Now suppose you leave your money in the bank for two years instead of one, at the same 5% per annual interest. It is the same as re-investing the \$105 for another single year. Using the same concept as above:

$$C_2 = 105 + 105(.05) = 105 + 5.25 = \$110.25$$

Again, using symbols instead of the specific numbers to obtain the more general relation:

$$C_2 = C_1(1 + r) \quad \text{but since } C_1 = C_0(1 + r), \text{ we can substitute for } C_1 \text{ and obtain:}$$

$$C_2 = C_0(1 + r)(1 + r) = C_0(1 + r)^2$$

Checking this formula with our specific example, we obtain the same as before:

$$C_2 = 100(1 + .05)^2 = \$110.25$$

We have captured with this simple two year example a lot of the essence of what is behind time value and return calculations in finance. What we are doing here is compounding, and we have just calculated the future value after two years of a cash flow at time $t = 0$ when the annual interest rate (compounded annually) is rate r . It is useful to decompose the previous general formula into more simple terms. Using the rules of algebra:

$$C_2 = C_0(1 + r)(1 + r) = C_0(1 + 2r + r^2) = C_0 + C_0 2r + C_0(r^2)$$

If we denote one year's interest (not rate but actual cash flow due to interest) as $I = C_0(r)$,

$$C_2 = C_0 + 2I + rI.$$

It is clear that our future cash flow at $t = 2$ consist of:

1. The return of the principal
2. Two simple interest payments, one for each of the two years
3. One year's interest on the first year interest.

It is this last factor which demonstrates the compounding effect of letting the interest accumulate and being paid interest on these funds. But compounding can occur in many other examples of finance, not just interest payments from savings accounts. It is a fairly important concept.

Checking this new formula with our numbers:

$$C_2 = 100 + 2(5) + (.05)(5) = 100 + 10 + .25 = \$110.25$$

If the interest on interest is ignored, we call it **simple interest**. In our example the difference is only 25 cents, but as the number of periods increases, this difference becomes much larger, because you start accumulating interest on interest on interest, etc.

An example of a longer investing periods would demonstrate the dramatic effect of compounding more effectively, but first we need a general formula to do this. This can be easily obtained if we consider the previous pattern:

$$C_1 = C_0(1 + r)$$

$$C_2 = C_0(1 + r)^2$$

In general,

$$C_n = C_0(1 + r)^n$$

We can call C_n the future value at $t = n$: **FV_n**. Therefore:

$$FV_n = C_0(1 + r)^n$$

This formula is also called the **future value formula**. It can easily be calculated using the power function key y^x on your calculator. Use it to verify the future value of \$100 at 5% interest compounded annually after 50 years:

$$FV_{50} = 100(1 + .05)^{50} = 100(1.05)^{50} = 100(11.4674) = \$1,146.74$$

Compare that to the simple interest calculation:

$$100 + 50(.05)(100) = \$350$$

The difference between the two numbers is clearly more than 50 times .25 (the amount of the difference for the two year example). $(50)(.25) = 12.50$

Compounding effect is particularly strong for long periods and high interest rates. Try this example with 10% interest.

Multiple Compounding per Year

Sometimes banks take a stated **nominal** interest rate, and divide this rate by a number m , and compound the interest m times per year. Common compounding periods per year include semi-annual ($m = 2$), monthly ($m = 12$), or daily ($m = 365$). The future value formula for multiple compounding per year is a simple extension of the annual compounding per year:²

$$F_n = C_0 [1 + (r/m)]^{(r)(m)}$$

Example: What is the future value of \$100 after two years, in a savings account at 5% nominal interest and *monthly* compounding?

Solution: $FV_2 = 100[1 + (.05/12)]^{(2)(12)} = 100(1 + 0.00416666)^{24} = 110.49$

Notice that this is greater than the 110.25 obtained under annual compounding. This is because of the compounding effect described above.

If banks and other institutions offer different rates and different compounding periods, it becomes difficult to compare rates. The **effective annual rate** or **EAR** is the rate you would obtain which would produce identical future values as the multiple compounding, but using annual compounding. This definition should allow you to set up an equation which allows you to solve for the r_{EAR} .

² An extreme form of compounding is called continuous compounding. The formula for future value is $FV_n = C_0 e^{(r)(n)}$, where e is the exponential function e^x found on your calculator. A simple example shows that results using continuous compounding are not that different from daily compounding.

Example: What is the EAR for the previous problem?

Solution: Write the equation for the future value for a single year two different ways and set them equal to each other. One way is with a single annual compounding period (the r_{EAR}). The other way is with the multiple compounding at rate r_{NOM}/m . With this action you have made the future value the same. There is no need to do it for multiple years, although the same answer would result but with more work. The initial cash flows, C_0 , cancel, and you can now solve for the EAR:

$$C_0(1 + r_{EAR}) = C_0 \left(1 + \frac{r_{NOM}}{m} \right)^m$$

$$r_{EAR} = \left(1 + \frac{r_{NOM}}{m} \right)^m - 1$$

$$r_{EAR} = \left(1 + \frac{.05}{12} \right)^{12} - 1 = .0511619$$

As expected, the EAR is greater than the 5%, indicating that you do better with multiple compounding.

Discounting and Present Value

An even more important concept than future value is **present value**. This is value today of cash flows which will occur in the future. Firms make financial decisions based on present value. If we know for certain what cash flows are associated with a financial asset, and we know what interest rate to use for our formulas, (both of which can be quite uncertain) we could easily value the asset. This tool is also useful for valuing stocks, bonds, and other assets purchased by investors. The present value formula is simply the inverse of the future value formula.

$$\text{If } C_0(1 + r)^n = FV_n$$

If we call the FV_n (a cash flow at time $t = n$): C_n , and call C_0 : **PV** (the present value of this future cash flow), we could solve this equation for the PV using algebra.³

$PV(1 + r)^n = C_n$ and solving for PV:

$$PV = C_n \left[\frac{1}{(1 + r)^n} \right] = C_n(1 + r)^{-n}$$

³ The use of negative exponents is explained in the appendix.

$(1+r)^{-n}$ is called the **discount factor** or the **discount function**. It is easily found using the power function on your calculator.

What is the present value of \$110.25, obtained two years from today if the interest rate is 5%?

$$PV = 110.25(1.05)^{-2} = 110.25(.90703) = \$100.00$$

You should not be surprised at this result!

Present Value Additivity Rule

Suppose you have a cash flow at two points in time such as $t = 1$ and $t = 2$. Dollars at different points in time are not equivalent because of the time value of money. To find out what the total value is worth, you can not add across different time periods. Instead, you need to take the present value of each cash flow, and then you may add. Cash flows at the same point in time may be added together. The present value of the sum of the above would be:

$$PV = PV \text{ of } C_1 \text{ and } PV \text{ of } C_2 = C_1(1+r)^{-1} + C_2(1+r)^{-2}$$

In general, the formula to discount any set of future cash flows can be written in this very compact manner:⁴

$$PV = \sum_{t=0}^n C_t(1+r)^{-t}$$

This is also known as the **present value additivity rule**.

What is the present value of \$10 obtained at $t = 1$ and \$10 obtained at $t = 2$ with $r = 10\%$?

$$PV = (10)(1+.10)^{-1} + (10)(1+.10)^{-2}$$

$$PV = (10)(.90909) + (10)(.82645)$$

$$PV = 9.0909 + 8.2645 = 17.36$$

Draw the cash flows represented by the above problem.

⁴ See the appendix for details on the summation notation.

Appendix: Review of the Mathematical Notation

A **negative exponent** is equivalent to the quantity one over the same number raised to the same but positive exponent.

$$\frac{1}{y^x} = y^{-x}$$

To convince yourself this is true, try the following example:

$$\frac{1}{2^3} = \frac{1}{8} = 0.125$$
$$2^{-3} = 0.125$$

The negative exponent is often used to save space, but you need to understand that it is more than just a notation device. You can actually enter a negative exponent on in your power function and get the correct answer, and this is usually the quickest way to perform discounting on your calculator. Try obtaining the following discount function value on your calculator using the power function and negating the exponent, rather than using the **1/x** key to invert.

$$(1.05)^{-2} = .907029479$$

The **summation sign** is used a lot in finance (and also statistics). It is the Greek capital letter for sigma (which is like our letter S and stands for sum). The variable after the summation sign takes on the value of the various terms, and they are all to be added. To differentiate between the terms, an index variable is used. To indicate how many terms there are, one scheme uses lower and upper limits of the subscripts, and puts them below and above the summation sign. For example:

$$\sum_{i=m}^n a_i = a_m + a_{m+1} + a_{m+2} + \dots a_n$$

In this example, there are multiple values of variable a . Index variable i appears as a subscript, and its only job, in this case, is to keep track of the terms. The first term will be term m and the last will be n . A concrete example of this is as follows:

Suppose $a_1 = 2$, $a_2 = -3$, and $a_3 = 4$, then:

$$\sum_{i=1}^3 a_i = 2 - 3 + 4 = 3$$

The index variable i simply indicates the subscripts 1, 2, and 3, and it could have been represented by any letter. For this reason it is sometimes called a dummy variable.

Sometimes the index variable can serve not to just differentiate different terms but to actually act on a term as a mathematical operation. For example, if the subscript is a superscript and acts as an exponent:

$$\sum_{i=1}^3 2^i = 2^1 + 2^2 + 2^3 = 14$$

This is actually how the summation sign is used in compounding and discounting equations:

$$PV = \sum_{t=0}^n C_t (1+r)^{-t}$$