Proof of The Fundamental Theorem of Calculus

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Let $f$ be a continuous function defined on an open interval $I$. Let $a \in I$, and let $A_f(x) = \int_a^x f(t)\ dt$ for $x \in I$. Then $A'_f(x) = f(x)$.

Proof: By the definition of derivative, $A'_f(x) = \lim_{h \to 0} \frac{A_f(x+h) - A_f(x)}{h}$.

Using the definition of the function $A_f$ and properties of the definite integral, we have

$$A_f(x+h) - A_f(x) = \int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt = \int_a^{x+h} f(t)\ dt + \int_x^a f(t)\ dt = \int_x^{x+h} f(t)\ dt.$$

Therefore, $A'_f(x) = \lim_{h \to 0} \left[ \frac{1}{h} \int_x^{x+h} f(t)\ dt \right]$.

Since $f$ is continuous on $I$ and since the interval $[x, x+h]$ is contained in $I$, the Extreme Value Theorem implies that $f$ achieves its minimum and maximum values on $[x, x+h]$. Let $m$ denote the minimum value of $f$ on $[x, x+h]$, and let $M$ denote the maximum value.

$m \le f(t) \le M$ for all $t \in [x, x+h]$ implies that $\int_x^{x+h} m\ dt \le \int_x^{x+h} f(t)\ dt \le \int_x^{x+h} M\ dt$.

It follows that $mh \le \int_x^{x+h} f(t)\ dt \le Mh$, so $m \le \bar{f} \le M$, where $\bar{f} = \frac{1}{h} \int_x^{x+h} f(t)\ dt$.

As $h \to 0$, the interval $[x, x+h]$ shrinks to the point $x$. Since $f$ is continuous on $[x, x+h]$, the minimum and maximum values of $f$ on $[x, x+h]$ must approach $f(x)$ as $h \to 0$.

Since $m \le \bar{f} \le M$, the Squeeze Theorem implies that $\lim_{h \to 0} \bar{f} = f(x)$.

Since $A'_f(x) = \lim_{h \to 0} \bar{f}$, it follows that $A'_f(x) = f(x)$. QED