92.445/545 Partial Differential Equations

Solution of the Laplace Equation by Separation of Variables

Below is a summary of the results developed in class on April 8.

1. The Laplace equation on a rectangle with Dirichlet boundary conditions.

   The solution of the boundary value problem

$$\begin{aligned} \Delta u &= 0 \text{ on } 0 < x < b,\ 0 < y < d \\ u(0,y) &= f(y)\ 0 < y < d \\ u(b,y) &= g(y)\ 0 < y < d \\ u(x,0) &= h(x)\ 0 < x < b \\ u(x,d) &= k(x)\ 0 < x < b \end{aligned}$$

   can be expressed as

$$\begin{aligned} u &= \sum_{n=1}^{\infty} \left\{ \sin\left(\frac{n\pi y}{d}\right)\left[A_n \sinh\left(\frac{n\pi x}{d}\right) + B_n \sinh\left(\frac{n\pi}{d}(x-b)\right)\right] \right. \\ &\quad \left. + \sin\left(\frac{n\pi x}{b}\right)\left[C_n \sinh\left(\frac{n\pi y}{b}\right) + D_n \sinh\left(\frac{n\pi}{b}(y-d)\right)\right]\right\} \end{aligned}$$

   where $A_n \sinh\left(\frac{n\pi b}{d}\right)$ equals the nth Fourier sine series coefficient of $g(y)$ on $0 < y < d$, $-B_n \sinh\left(\frac{n\pi b}{d}\right)$ equals the nth Fourier sine series coefficient of $f(y)$ on $0 < y < d$, $C_n \sinh\left(\frac{n\pi d}{b}\right)$ equals the nth Fourier sine series coefficient of $k(x)$ on $0 < x < b$, and $-D_n \sinh\left(\frac{n\pi d}{b}\right)$ equals the nth Fourier sine series coefficient of $h(x)$ on $0 < x < b$.

2. The Laplace equation on a disk with Dirichlet boundary conditions.

   The solution of the boundary value problem

$$\begin{aligned} \Delta w &= 0 \text{ on } 0 < r < a,\ 0 \le \theta \le 2\pi \\ w(a,\theta) &= h(\theta)\ 0 \le \theta \le 2\pi \end{aligned}$$

   can be expressed as

$$w(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left[A_n \cos(n\theta) + B_n \sin(n\theta)\right]$$

   where $a^n A_n$ is the nth Fourier cosine series coefficient of $h(\theta)$ on $0 \le \theta \le 2\pi$ and $a^n B_n$ is the nth Fourier sine series coefficient of $h(\theta)$ on $0 \le \theta \le 2\pi$.