92.445/545 Partial Differential Equations

Classification of Second Order Linear PDE's and Reduction to Canonical Form

A second order pde in 2 independent variables is *linear* if it can be written in the form

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y) \tag{1}$$

This pde is said to be *hyperbolic* at the point (x,y) if $b^2 - ac > 0$, *parabolic* at (x,y) if $b^2 - ac = 0$, or *elliptic* at (x,y) if $b^2 - ac < 0$.

The pde is hyperbolic (or parabolic or elliptic) on a region D if the pde is hyperbolic (or parabolic or elliptic) at each point of D.

A second order linear pde can be reduced to so-called canonical form by an appropriate change of variables $\xi = \xi(x,y),\ \eta = \eta(x,y)$.

The *Jacobian* of this transformation is defined to be $J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x\eta_y - \eta_x\xi_y$.

The Jacobian should be nonzero to ensure that the transformation is invertible. In that case, we can, at least in principle, solve for x and y as functions of ξ and η. We let $w(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta))$.

Using the Chain Rule, one can show that equation (1) takes the following form when expressed in terms of the variables ξ and η:

$$Aw_{\xi\xi} + 2Bw_{\xi\eta} + Cw_{\eta\eta} + Dw_\xi + Ew_\eta + Fw = G(\xi,\eta) \tag{2}$$

where

$$\begin{aligned}
A &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 \\
B &= a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + c\xi_y\eta_y \\
C &= a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 \\
D &= a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y \\
E &= a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy} + d\eta_x + e\eta_y \\
F &= f(x(\xi,\eta), y(\xi,\eta)) \\
G &= g(x(\xi,\eta), y(\xi,\eta))
\end{aligned}$$

As shown in Pinchover & Rubinstein's book, the type of the equation is not affected by the change of variables. If equation (1) is hyperbolic (or parabolic, or elliptic) at the point (x,y), then equation (2) is also hyperbolic (or parabolic, or elliptic) at the point (ξ,η).

Note that the expressions for A and C can be factored:

$$A = \frac{1}{a}\left[a\xi_x + (b + \sqrt{b^2 - ac})\xi_y\right]\left[a\xi_x + (b - \sqrt{b^2 - ac})\xi_y\right] \tag{3}$$

$$C = \frac{1}{a}\left[a\eta_x + (b + \sqrt{b^2 - ac})\eta_y\right]\left[a\eta_x - (b + \sqrt{b^2 - ac})\eta_y\right] \tag{4}$$

1. **Hyperbolic Equations**

The canonical form of a hyperbolic equation is

$$w_{\xi\eta} + \hat{D} w_\xi + \hat{E} w_\eta + \hat{F} w = \hat{G}(\xi, \eta) \tag{5}$$

The canonical variables ξ and η for a hyperbolic pde satisfy the equations

$$a\xi_x + \left(b + \sqrt{b^2 - ac}\right)\xi_y = 0 \tag{6}$$

and

$$a\eta_x + \left(b - \sqrt{b^2 - ac}\right)\eta_y = 0 \tag{7}$$

making coefficients A and C in (2) zero by virtue of (3) and (4).

The families of curves $\xi =$ constant and $\eta =$ constant are the characteristic curves. **Hyperbolic equations have two families of characteristic curves.**

Example. Consider the pde $u_{xx} + 4u_{xy} + u_x = 0$. In this problem, $a = 1$, $2b = 4$, and $c = 0$, so $b^2 - ac = 2^2 - (1)(0) = 4 > 0$, and the given pde is hyperbolic on the entire xy plane. Equations (6) and (7) reduce to $\xi_x + 4\xi_y = 0$ and $\eta_x = 0$. Solving these equations by the method of characteristics, we find that $\xi = f(4x - y)$ and $\eta = g(y)$. For simplicity we take $\boxed{\xi = 4x - y}$ and $\boxed{\eta = y}$. We therefore have

$$\begin{aligned} u_x &= w_\xi \xi_x + w_\eta \eta_x = 4w_\xi \\ u_{xx} &= 4\left[w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x\right] = 16w_{\xi\xi} \\ u_{xy} &= 4\left[w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y\right] = -4w_{\xi\xi} + 4w_{\xi\eta} \end{aligned}$$

Therefore, the given pde $u_{xx} + 4u_{xy} + u_x = 0$ becomes

$[16w_{\xi\xi}] + 4\left[-4w_{\xi\xi} + 4w_{\xi\eta}\right] + [4w_\xi] = 0$, or $16w_{\xi\eta} + 4w_\xi = 0$, or $\boxed{w_{\xi\eta} + \frac{1}{4}w_\xi = 0}$.

2. **Parabolic Equations**

The canonical form of a parabolic equation is

$$w_{\xi\xi} + \hat{D} w_\xi + \hat{E} w_\eta + \hat{F} w = \hat{G}(\xi, \eta) \tag{8}$$

For a parabolic equation, $b^2 - ac = 0$ so equations (3) and (4) reduce to the same equation:

$$A = \frac{1}{a}\left[a\xi_x + b\xi_y\right]^2 \tag{9}$$

$$C = \frac{1}{a}\left[a\eta_x + b\eta_y\right]^2 \tag{10}$$

Instead of two equations like (6) and (7) for hyperbolic equations, we have just the single equation $a\xi_x + b\xi_y = 0$ (or $a\eta_x + b\eta_y = 0$). **Parabolic equations have only one family of characteristic curves.**

We choose the canonical variable η to be a solution of the equation

$$a\eta_x + b\eta_y = 0 \tag{11}$$

and we choose ξ to be any function which makes the Jacobian $\xi_x\eta_y - \xi_y\eta_x$ nonzero. The choice of η makes $C = 0$. Because $B^2 - AC = 0$, that makes $B = 0$ and therefore the only nonzero second derivative term in the pde (2) is $Aw_{\xi\xi}$.

Example. Consider the pde $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0$ for $x > 0$. (Pinchover & Rubinstein p. 70). In this problem, $a = x^2$, $b = -xy$, and $c = y^2$ so $b^2 - ac = (-xy)^2 - x^2y^2 = 0$ and the given pde is parabolic on the half-plane $x > 0$. Equation (11) becomes $x^2\eta_x - xy\eta_y = 0$, or $x\eta_x - y\eta_y = 0$. Using the method of characteristics, we find that $\eta = f(xy)$. For simplicity we take $\eta = xy$. If we just take $\xi = x$, the Jacobian of the transformation becomes $\xi_x\eta_y - \xi_y\eta_x = (1)(x) - (0)(y) = x > 0$. We can therefore take $\boxed{\xi = x}$ and $\boxed{\eta = xy}$. With this choice, we obtain

$$
\begin{aligned}
u_x &= w_\xi\xi_x + w_\eta\eta_x = w_\xi + yw_\eta \\
u_y &= w_\xi\xi_y + w_\eta\eta_y = 0 \cdot w_\xi + xw_\eta = xw_\eta \\
u_{xx} &= [w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x] + y\,[w_{\eta\xi}\xi_x + w_{\eta\eta}\eta_x] = w_{\xi\xi} + 2yw_{\xi\eta} + y^2w_{\eta\eta} \\
u_{xy} &= [w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y] + \underbrace{w_\eta + y\,[w_{\eta\xi}\xi_y + w_{\eta\eta}\eta_y]}_{\text{Product Rule}} = w_\eta + xw_{\xi\eta} + xyw_{\eta\eta} \\
u_{yy} &= x\,[w_{\eta\xi}\xi_y + w_{\eta\eta}\eta_y] = x^2w_{\eta\eta}
\end{aligned}
$$

Therefore, the given pde $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0$ becomes
$x^2\left[w_{\xi\xi} + 2yw_{\xi\eta} + y^2w_{\eta\eta}\right] - 2xy\,[w_\eta + xw_{\xi\eta} + xyw_{\eta\eta}] + y^2\left[x^2w_{\eta\eta}\right] + x\,[w_\xi + yw_\eta] + y\,[xw_\eta] = 0$,
or $x^2w_{\xi\xi} + xw_\xi = 0$ or $\boxed{w_{\xi\xi} + \frac{1}{\xi}w_\xi = 0}$. (Here we have used the fact that $\xi = x$.)

3. **Elliptic Equations**

The canonical form of an elliptic equation is

$$w_{\xi\xi} + w_{\eta\eta} + \hat{D}w_\xi + \hat{E}w_\eta + \hat{F}w = \hat{G}(\xi, \eta) \tag{12}$$

For an elliptic equation, $b^2 - ac < 0$ so equations (3) and (4) contain complex coefficients and have no real solutions. **Elliptic equations have no characteristic curves.**

In order for (2) to reduce to (12), we must have $A = C$ and $B = 0$, or $A - C = 0$ and $B = 0$:

$$
\begin{aligned}
a\left(\xi_x^2 - \eta_x^2\right) + 2b\,(\xi_x\xi_y - \eta_x\eta_y) + c\left(\xi_y^2 - \eta_y^2\right) &= 0 \text{ and} \\
a\xi_x\eta_x + b\,(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y &= 0
\end{aligned}
$$

Adding the first of these equations to i times the second, we obtain

$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = 0 \tag{13}$$

where $\phi = \xi + i\eta$. Factoring equation (13), we obtain

$$\frac{1}{a}\left[a\phi_x + \left(b + i\sqrt{ac - b^2}\right)\phi_y\right]\left[a\phi_x + \left(b - i\sqrt{ac - b^2}\right)\phi_y\right] = 0 \tag{14}$$

We will take ϕ to be the solution of

$$a\phi_x + \left(b + i\sqrt{ac - b^2}\right)\phi_y = 0 \tag{15}$$

and then we will use the change of variables given by $\xi = \Re(\phi)$ and $\eta = \Im(\phi)$.

Example. Consider the pde $u_{xx} + xu_{yy} = 0$ for $x > 0$. (Pinchover & Rubinstein p. 72). In this problem, $a = 1$, $b = 0$, and $c = x$ so $b^2 - ac = 0^2 - (1)(x) = -x < 0$ and the given pde is elliptic on the half-plane $x > 0$. Equation (15) becomes $\phi_x + i\sqrt{x}\phi_y = 0$. We take the initial data curve to be the x axis, so the intial curve Γ can be parameterized as $x = s$, $y = 0$. The characteristic curves satisfy the conditions $\dfrac{dx}{dt} = 1$ and $\dfrac{dy}{dt} = i\sqrt{x}$. $\dfrac{dx}{dt} = 1 \Rightarrow x = t + f(s)$. Because $x = s$ on Γ (where $t = 0$), $f(s)$ must equal s. Therefore, $x = t + s$ and $\dfrac{dy}{dt} = i\sqrt{x} = i\sqrt{t+s} \Rightarrow y = i\dfrac{2}{3}(t+s)^{3/2} + g(s)$.

Because $y = 0$ on Γ (where $t = 0$), $g(s)$ must equal $-i\frac{2}{3}s^{3/2}$. Therefore, $y = i\dfrac{2}{3}(t+s)^{3/2} - i\dfrac{2}{3}s^{3/2}$

$\Rightarrow y = \dfrac{2i}{3}x^{3/2} - \dfrac{2i}{3}s^{3/2} \Rightarrow s^{3/2} = x^{3/2} + i\left(\dfrac{3y}{2}\right)$.

On characteristics, $\dfrac{d\phi}{dt} = \dfrac{\partial\phi}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial\phi}{\partial y}\dfrac{dy}{dt} = \phi_x \cdot 1 + \phi_y \cdot i\sqrt{x} = \phi_x + i\sqrt{x}\phi_y = 0$ from the given pde. $\dfrac{d\phi}{dt} = 0 \Rightarrow \phi = h(s)$. For simplicity we take $h(s) = s^{3/2}$. Therefore, $\phi = s^{3/2} = x^{3/2} + i\left(\dfrac{3y}{2}\right)$ so $\boxed{\xi = \Re(\phi) = x^{3/2}}$ and $\boxed{\eta = \Im(\phi) = \dfrac{3y}{2}}$. With this choice, we obtain

$$
\begin{aligned}
u_x &= w_\xi \xi_x + w_\eta \eta_x = \frac{3}{2}x^{1/2}w_\xi + w_\eta \cdot 0 = \frac{3}{2}x^{1/2}w_\xi \\
u_y &= w_\xi \xi_y + w_\eta \eta_y = 0 \cdot w_\xi + \frac{3}{2}w_\eta = \frac{3}{2}w_\eta \\
u_{xx} &= \frac{3}{4}x^{-1/2}w_\xi + \frac{3}{2}x^{1/2}\left[w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x\right] = \frac{3}{4}x^{-1/2}w_\xi + \frac{3}{2}x^{1/2}\left[\frac{3}{2}x^{1/2}w_{\xi\xi}\right] = \frac{3}{4}x^{-1/2}w_\xi + \frac{9x}{4}w_{\xi\xi} \\
u_{yy} &= \frac{3}{2}\left[w_{\eta\xi}\xi_y + w_{\eta\eta}\eta_y\right] = \frac{3}{2}\left[\frac{3}{2}w_{\eta\eta}\right] = \frac{9}{4}w_{\eta\eta}
\end{aligned}
$$

Therefore, the given pde $u_{xx} + xu_{yy} = 0$ becomes $\left[\dfrac{3}{4}x^{-1/2}w_\xi + \dfrac{9x}{4}w_{\xi\xi}\right] + x\left[\dfrac{9}{4}w_{\eta\eta}\right] = 0$, or $9x\left[w_{\xi\xi} + w_{\eta\eta}\right] + 3x^{-1/2}w_\xi = 0$, or $w_{\xi\xi} + w_{\eta\eta} + \dfrac{1}{3x^{3/2}}w_\xi = 0$, or $\boxed{w_{\xi\xi} + w_{\eta\eta} + \dfrac{1}{3\xi}w_\xi = 0.}$

Here we have used the fact that $\xi = x^{3/2}$.