

Final Exam

Due May 13 - No Extensions

PLEASE SHOW ALL WORK! You will not receive full credit if you do not show your work. Citing *Mathematica* or an online or hard copy reference (except for evaluating integrals) does not constitute showing your work.

You must do your own work. Please do not consult anyone other than me regarding this exam. Violations will result in a grade of 0 for this exam.

Problem 1. (10 points) Consider the pde $yu_{xx} + xu_{yy} + u_x - u_y = x^2 + y^2$

- In what region(s) of the xy plane is this pde elliptic?
- In what region(s) of the xy plane is this pde hyperbolic?

Problem 2. (20 points)

Solve the Cauchy problem $u_x + 2xu_y = 8xy$, $u(0, y) = \sin(y)$

Problem 3. (20 points)

Solve the following Cauchy problem for the wave equation on a semi-infinite domain.

$$\begin{aligned}u_{tt} - 9u_{xx} &= 0 && \text{on } 0 < x < \infty, t > 0 \\u(x, 0) &= e^{-x} && \text{on } 0 < x < \infty \\u_t(x, 0) &= \cos(x) && \text{on } 0 < x < \infty \\u_x(0, t) &= 0 && \text{on } t > 0\end{aligned}$$

Problem 4. (20 points)

Solve the following IBVP for the heat equation with nonzero boundary conditions.

$$\begin{aligned}u_t &= u_{xx} && \text{on } 0 < x < 1, t > 0 \\u(0, t) &= 1 && t \geq 0 \\u(1, t) &= 2 && t \geq 0 \\u(x, 0) &= x^2 + 1 && 0 \leq x \leq 1\end{aligned}$$

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Problem 5. (15 points)

Solve the following BVP for the Laplace equation on a rectangle.

$$\begin{aligned} \Delta u &= 0 && \text{on } 0 < x < \pi, 0 < y < \pi \\ u(x, 0) &= 0 && \text{on } 0 < x < \pi \\ u(x, \pi) &= 0 && \text{on } 0 < x < \pi \\ u(0, y) &= 0 && \text{on } 0 < y < \pi \\ u(\pi, y) &= \sin(3y) && \text{on } 0 < y < \pi \end{aligned}$$

Problem 6. (15 points)

Use a Green's function to solve Poisson's equation $\Delta u = x$ in the right half-plane $\{(x, y) | x > 0\}$ with boundary condition $u(0, y) = 0$. You will need to use the reflection principle to find the Green's function. You may leave your solution in the form of an integral.

Extra Credit (10 points)

As we previously discussed, if u is a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ on $-\infty < x < \infty, t > 0$ for which $u \rightarrow 0$ and $u_x \rightarrow 0$ as $x \rightarrow \pm\infty$, then the energy $E = \int_{-\infty}^{\infty} (u_t^2 + c^2 u_x^2) dx$ is constant.

Can you find a corresponding conserved quantity \hat{E} for solutions of the heat equation $u_t = k u_{xx}$ on $-\infty < x < \infty, t > 0$ assuming $u \rightarrow 0$ and $u_x \rightarrow 0$ as $x \rightarrow \pm\infty$?

FOR STUDENTS ENROLLED IN 92.545.

Problem 7. (20 points) (Pinchover & Rubinstein problem 7.11) Solve the Laplace equation $\Delta u = 0$ in the domain $x^2 + y^2 > 4$ subject to the boundary condition $u(x, y) = y$ on $x^2 + y^2 = 4$ and the decay condition $u(x, y) \rightarrow 0$ as $|x| + |y| \rightarrow \infty$.

Hint: Recall that in deriving (7.65) on p. 197 of Pinchover & Rubinstein we discarded the r^{-n} term in (7.63) because we were looking for a solution that was bounded on the disk

$B_a = \{(r, \theta) | 0 < r < a, 0 \leq \theta \leq 2\pi\}$. In this problem you need to keep the r^{-n} term but discard the r^n term because $r^n \rightarrow \infty$ as $r \rightarrow \infty$.