92.445/545 Partial Differential Equations   Spring 2013

Final Exam Solutions

Problem 1. (10 points) Consider the pde $yu_{xx} + xu_{yy} + u_x - u_y = x^2 + y^2$

a) In what region(s) of the xy plane is this pde elliptic?

Recall that the second order linear pde in 2 independent variables $au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$ is hyperbolic if $b^2 - ac > 0$, parabolic if $b^2 - ac = 0$, and elliptic if $b^2 - ac < 0$. (See page 65 of Pinchover & Rubinstein.)

Here $a = y$, $2b = 0$, and $c = x$, so $b^2 - ac = -xy$. $-xy < 0$ where $xy > 0$. Therefore, $\boxed{\text{the given pde is elliptic in the first and third quadrants.}}$

b) In what region(s) of the xy plane is this pde hyperbolic?

$b^2 - ac = -xy > 0$ where $xy < 0$. Therefore, $\boxed{\text{the given pde is hyperbolic in the second and fourth quadrants.}}$

Problem 2. (20 points)

Solve the Cauchy problem $u_x + 2xu_y = 8xy,\ u(0, y) = \sin(y)$

Recall that the second order linear pde in 2 independent variables $au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$ is hyperbolic if $b^2 - ac > 0$, parabolic if $b^2 - ac = 0$, and elliptic if $b^2 - ac < 0$. (See page 65 of the textbook.)

The initial curve Γ is given by $x = 0, y = s$. On Γ we have $u(s) = \sin(s)$. The characteristic curves satisfy the conditions $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = 2x$. $\frac{dx}{dt} = 1 \Rightarrow x = t + c_1(s)$. Because $x = 0$ on Γ (where $t = 0$), $c_1(s) = 0$, so $x = t$. $\frac{dy}{dt} = 2x = 2t \Rightarrow y = t^2 + c_2(s)$. Because $y = s$ on Γ (where $t = 0$), $c_2(s) = s$, so $y = t^2 + s$. On characteristics, $\frac{du}{dt} = 8xy$ from the given pde.

$\frac{du}{dt} = 8xy = 8t\left(t^2 + s\right) = 8t^3 + 8st \Rightarrow u = \int \left(8t^3 + 8st\right) dt \Rightarrow u = 2t^4 + 4st^2 + c_3(s)$

Because $u = \sin(s)$ on Γ (where $t = 0$), $c_3(s)$ must equal $\sin(s)$, so $u = 2t^4 + 4st^2 + \sin(s)$.

$x = t$ and $y = t^2 + s \Rightarrow t = x$ and $s = y - x^2$.

Therefore, $u = 2t^4 + 4st^2 + \sin(s) = 2x^4 + 4\left(y - x^2\right)x^2 + \sin\left(y - x^2\right) \Rightarrow \boxed{u(x, y) = 4x^2y - 2x^4 + \sin\left(y - x^2\right).}$

Problem 3. (20 points)

Solve the following Cauchy problem for the wave equation on a semi-infinite domain.

$$\begin{aligned} u_{tt} - 9u_{xx} &= 0 \quad \text{on } 0 < x < \infty, t > 0 \\ u(x, 0) &= e^{-x} \text{ on } 0 < x < \infty \\ u_t(x, 0) &= \cos(x) \text{ on } 0 < x < \infty \\ u_x(0, t) &= 0 \text{ on } t > 0 \end{aligned}$$

As we discussed in class, the solution to the Cauchy problem

$$\begin{aligned} u_{tt} - c^2u_{xx} &= 0 \quad \text{on } 0 < x < \infty, t > 0 \\ u(x, 0) &= f(x) \\ u_t(x, 0) &= g(x) \\ u_x(0, t) &= 0 \end{aligned}$$

equals the solution of the Cauchy problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \qquad \text{on } -\infty < x < \infty, t > 0 \\ u(x,0) &= \hat{f}(x) \\ u_t(x,0) &= \hat{g}(x) \end{aligned}$$

restricted to the interval $0 < x < \infty$. Here $\hat{f}$ and $\hat{g}$ are the functions obtained by extending f and g, respectively, to the entire real line as even functions. In this problem, $f(x) = e^{-x}$ and $g(x) = \cos(x)$. Their extensions as even functions are $\hat{f}(x) = e^{-|x|}$ and $\hat{g}(x) = \cos(x)$. In this problem, $c = 3$, so D'Alembert's Formula gives us

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\hat{f}(x+ct) + \hat{f}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \hat{g}(s)\, ds \\ &= \frac{1}{2}\left[e^{-|x+3t|} + e^{-|x-3t|}\right] + \frac{1}{2\cdot 3}\int_{x-3t}^{x+3t} \cos(s)\, ds \\ &= \frac{1}{2}\left[e^{-|x+3t|} + e^{-|x-3t|}\right] + \frac{1}{6}\left[\sin(s)|_{x-3t}^{x+3t}\right] \end{aligned}$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\left[e^{-|x+3t|} + e^{-|x-3t|}\right] + \frac{1}{6}\left[\sin(x+3t) - \sin(x-3t)\right]}$$

Problem 4. (20 points)

Solve the following IBVP for the heat equation with nonzero boundary conditions.

$$\begin{aligned} u_t &= u_{xx} \qquad \text{on } 0 < x < 1, t > 0 \\ u(0,t) &= 1 \quad t \geq 0 \\ u(1,t) &= 2 \quad t \geq 0 \\ u(x,0) &= x^2 + 1 \quad 0 \leq x \leq 1 \end{aligned}$$

As discussed in class, we look for a solution in the form $u(x,t) = u_1(x) + u_2(x,t)$ where $u_1(x)$ is a linear function of x satisfying the boundary conditions and $u_2(x,t)$ is a solution of the IBVP

$$\begin{aligned} u_{2t} &= u_{2xx} \qquad \text{on } 0 < x < 1, t > 0 \\ u(0,t) &= 0 \quad t \geq 0 \\ u(\pi,t) &= 0 \quad t \geq 0 \\ u(x,0) &= x^2 + 1 - u_1(x) \quad 0 \leq x \leq 1 \end{aligned}$$

In this problem, because the boundary conditions are $u(0,t) = 1$ and $u(1,t) = 2$, we have $u_1(x) = x + 1$. Therefore, u_2 is a solution of the IBVP

$$\begin{aligned} u_{2t} &= u_{2xx} \qquad \text{on } 0 < x < 1, t > 0 \\ u_2(0,t) &= 0 \quad t \geq 0 \\ u_2(\pi,t) &= 0 \quad t \geq 0 \\ u_2(x,0) &= x^2 - x \quad 0 \leq x \leq 1 \end{aligned}$$

As discussed in class and as described on pages 99 - 104 of Pinchover and Rubinstein, the solution of this IBVP can be expressed in the form $u_2(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-kn^2\pi^2 t/L^2} = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$.

$u_2(x,0) = x^2 - x \Rightarrow \sum_{n=1}^{\infty} B_n \sin(n\pi x) e^0 = x^2 - x \Rightarrow B_n =$ the nth Fourier sine series coefficient of $x^2 - x$ on $0 \le x \le 1 \Rightarrow B_n = \frac{2}{1}\int_0^1 \left(x^2 - x\right)\sin(n\pi x)\, dx = \begin{cases} -\frac{8}{n^3\pi^3} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

Therefore, $u(x,t) = u_1(x) + u_2(x,t) \Rightarrow \boxed{u(x,t) = x + 1 - \frac{8}{\pi^3}\sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin\left((2m-1)\pi x\right) e^{-(2m-1)^2\pi^2 t}}$

Problem 5. (15 points)

Solve the following BVP for the Laplace equation on a rectangle.

$$\begin{aligned} \Delta u &= 0 \quad \text{on } 0 < x < \pi, 0 < y < \pi \\ u(x,0) &= 0 \quad \text{on } 0 < x < \pi \\ u(x,\pi) &= 0 \quad \text{on } 0 < x < \pi \\ u(0,y) &= 0 \quad \text{on } 0 < y < \pi \\ u(\pi,y) &= \sin(3y) \quad \text{on } 0 < y < \pi \end{aligned}$$

As stated on the handout on solving the Laplace equation via separation of variables, the solution of the boundary value problem

$$\begin{aligned} \Delta u &= 0 \text{ on } 0 < x < b,\ 0 < y < d \\ u(0,y) &= f(y)\ 0 < y < d \\ u(b,y) &= g(y)\ 0 < y < d \\ u(x,0) &= h(x)\ 0 < x < b \\ u(x,d) &= k(x)\ 0 < x < b \end{aligned}$$

can be expressed as

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty} \left\{ \sin\left(\frac{n\pi y}{d}\right)\left[A_n \sinh\left(\frac{n\pi x}{d}\right) + B_n \sinh\left(\frac{n\pi(x-b)}{d}\right)\right] \right. \\ &\quad \left. + \sin\left(\frac{n\pi x}{b}\right)\left[C_n \sinh\left(\frac{n\pi y}{b}\right) + D_n \sinh\left(\frac{n\pi(y-d)}{b}\right)\right] \right\} \end{aligned}$$

where $A_n \sinh\left(\frac{n\pi b}{d}\right)$ equals the nth Fourier sine series coefficient of $g(y)$ on $0 < y < d$, $-B_n \sinh\left(\frac{n\pi b}{d}\right)$ equals the nth Fourier sine series coefficient of $f(y)$ on $0 < y < d$, $C_n \sinh\left(\frac{n\pi d}{b}\right)$ equals the nth Fourier sine series coefficient of $k(x)$ on $0 < x < b$, and $-D_n \sinh\left(\frac{n\pi d}{b}\right)$ equals the nth Fourier sine series coefficient of $h(x)$ on $0 < x < b$.

In this problem, $b = d = \pi$, so

$$u(x,y) = \sum_{n=1}^{\infty} \left\{\sin(ny)\left[A_n \sinh(nx) + B_n \sinh(n(x-\pi))\right] + \sin(nx)\left[C_n \sinh(ny) + D_n \sinh(n(y-\pi))\right]\right\}$$

$u(x,0) = 0 \Rightarrow$

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} \left\{\sin(0)\left[A_n \sinh(nx) + B_n \sinh(n(x-\pi))\right] + \sin(nx)\left[C_n \sinh(0) + D_n \sinh(n(0-\pi))\right]\right\} \\ &= \sum_{n=1}^{\infty} \sin(nx)\left[-D_n \sinh(n\pi)\right] \end{aligned}$$

$\Rightarrow D_n = 0$ for all n

$u(x, \pi) = 0 \Rightarrow$

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} \{\sin(n\pi)[A_n \sinh(nx) + B_n \sinh(n(x-\pi))] + \sin(nx)[C_n \sinh(n\pi) + D_n \sinh(n(\pi-\pi))]\} \\ &= \sum_{n=1}^{\infty} \sin(nx)[C_n \sinh(n\pi)] \end{aligned}$$

$\Rightarrow C_n = 0$ for all n

$u(0, y) = 0 \Rightarrow$

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} \{\sin(ny)[A_n \sinh(0) + B_n \sinh(n(0-\pi))] + \sin(0)[C_n \sinh(ny) + D_n \sinh(n(y-\pi))]\} \\ &= \sum_{n=1}^{\infty} \sin(ny)[-B_n \sinh(n\pi))] \end{aligned}$$

$\Rightarrow B_n = 0$ for all n

$u(\pi, y) = \sin(3y) \Rightarrow$

$$\begin{aligned} \sin(3y) &= \sum_{n=1}^{\infty} \{\sin(ny)[A_n \sinh(n\pi) + B_n \sinh(n(\pi-\pi))] + \sin(n\pi)[C_n \sinh(ny) + D_n \sinh(n(y-\pi))]\} \\ &= \sum_{n=1}^{\infty} \sin(ny)[A_n \sinh(n\pi))] \end{aligned}$$

$\Rightarrow A_3 = \dfrac{1}{\sinh(3\pi)}, \quad A_n = 0$ for $n \neq 3$

Therefore, $\boxed{u(x, y) = \dfrac{1}{\sinh(3\pi)} \sin(3y) \sinh(3x)}$

Problem 6. (15 points)

Use a Green's function to solve Poisson's equation $\Delta u = x$ in the right half-plane $\{(x, y)|x > 0\}$ with boundary condition $u(0, y) = 0$. You will need to use the reflection prinicple to find the Green's function. You may leave your solution in the form of an integral.

As discussed in class, the solution of the BVP

$$\Delta u = F(x, y) \text{ on } R, \quad u = 0 \text{ on } \partial R$$

is given by $u(x, y) = \int\int_R F(\xi, \eta) G(x, y; \xi, \eta)\, d\xi d\eta$, where the Green's function G satisfies the conditions

$$\Delta G = \delta(x - \xi, y - \eta) \text{ on } R, \quad G = 0 \text{ on } \partial R.$$

The function $G(x, y; \xi, \eta) = \dfrac{1}{4\pi} \ln\left[(x - \xi)^2 + (y - \eta)^2\right]$ satisfies $\Delta G = \delta(x - \xi, y - \eta)$ but not the boundary condition $G(0, y) = 0$. To find a Green's function that satisfies the boundary condition, we use the reflection principle and take

$$G(x,y;\xi,\eta) = \frac{1}{4\pi}\ln\left[(x-\xi)^2+(y-\eta)^2\right] - \frac{1}{4\pi}\ln\left[(x+\xi)^2+(y-\eta)^2\right] = \frac{1}{4\pi}\ln\left[\frac{(x-\xi)^2+(y-\eta)^2}{(x+\xi)^2+(y-\eta)^2}\right].$$

Since $F(x,y) = x$, the solution of the given BVP for Poisson's equation is

$$\boxed{u(x,y) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} \xi \ln\left[\frac{(x-\xi)^2+(y-\eta)^2}{(x+\xi)^2+(y-\eta)^2}\right]\, d\xi d\eta}$$

Extra Credit (10 points)

As we previously discussed, if u is a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ on $-\infty < x < \infty, t > 0$ for which $u \to 0$ and $u_x \to 0$ as $x \to \pm\infty$, then the energy $E = \int_{-\infty}^{\infty} \left(u_t^2 + c^2 u_x^2\right)\, dx$ is constant.

Can you find a corresponding conserved quantity $\hat{E}$ for solutions of the heat equation $u_t = k u_{xx}$ on $-\infty < x < \infty, t > 0$ assuming $u \to 0$ and $u_x \to 0$ as $x \to \pm\infty$?

Let $\hat{E} = \int_{-\infty}^{\infty} u\, dx$. Then

$$\frac{d\hat{E}}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} u\, dx = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}\, dx = \int_{-\infty}^{\infty} k u_{xx}\, dx \text{ (by the heat equation)} = k u_x\big|_{-\infty}^{\infty} = 0$$

because $u_x \to 0$ as $|x| \to \infty$. Therefore, $\hat{E}$ is constant.

FOR STUDENTS ENROLLED IN 92.545.

Problem 7. (20 points) (Pinchover & Rubinstein problem 7.11) Solve the Laplace equation $\Delta u = 0$ in the domain $x^2 + y^2 > 4$ subject to the boundary condition $u(x,y) = y$ on $x^2 + y^2 = 4$ and the decay condition $u(x,y) \to 0$ as $|x| + |y| \to \infty$.

Hint: Recall that in deriving (7.65) on p. 197 of Pinchover & Rubinstein we discarded the r^{-n} term in (7.63) because we were looking for a solution that was bounded on the disk $B_a = \{(r,\theta)|0 < r < a, 0 \le \theta \le 2\pi\}$. In this problem you need to keep the r^{-n} term but discard the r^n term because $r^n \to \infty$ as $r \to \infty$.

$$w(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n}\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right]$$

On the boundary $r = 2$ we have $w = y = 2\sin(\theta)$

Therefore, $2\sin(\theta) = A_0 + \sum_{n=1}^{\infty} (2)^{-n}\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right]$

$\Rightarrow A_n = 0\ (n \ge 0),\ 2^{-1}B_1 = 2,\ B_n = 0\ (n \ge 2).$

It follows that $w(r,\theta) = 4r^{-1}\sin(\theta) = 4r^{-2}r\sin(\theta)$, so $\boxed{u(x,y) = \frac{4y}{x^2+y^2}}$