92.445/545 Partial Differential Equations Spring 2013
Midterm Exam
Due March 18

PLEASE SHOW ALL WORK! You will not receive full credit if you do not show your work.
You must do your own work. Please do not consult anyone other than me regarding this exam. Violations will result in a grade of 0 for this exam.

Problem 1. (10 points)

Is the function given by $u(x,y) = x^2 + y^2$ a solution of the pde $yu_x - xu_y = 0$? Why or why not?

Problem 2. (30 points)

Solve the equation $\frac{1}{x}u_x - \frac{1}{y}u_y = 2u$ on $x > 0, y > 0$ with the initial condition $u(x,x) = x^2$.

Problem 3. (30 points) (Pinchover and Rubinstein problem 4.9)

Solve the following Cauchy problem for the nonhomogeneous wave equation.

$$\begin{aligned} u_{tt} - u_{xx} &= 1 \qquad \text{on } -\infty < x < \infty,\ t > 0 \\ u(x,0) &= x^2 \\ u_t(x,0) &= 1 \end{aligned}$$

Problem 4. (15 points) Classify each of the following pde's as hyperbolic, elliptic, or parabolic.

(a) $u_{xx} + 2u_{xy} + u_{yy} + u_x + u_y = 0$

(b) $u_{xx} + 2u_{xy} + 2u_{yy} + u_x + u_y = \sin(xy)$

(c) $2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 0$

Problem 5. (15 points) Find the canonical form of the following hyperbolic pde. Be sure to show the change of coordinates that reduces the pde to canonical form.

$$u_{xx} + 6u_{xy} - 16u_{yy} = 0$$

Extra Credit (10 points)

As mentioned in class, if u is a solution of the wave equation $u_{tt} - c^2u_{xx} = 0$ on $-\infty < x < \infty, t > 0$ for which $u \to 0, u_x \to 0$, and $u_t \to 0$ as $x \to \pm\infty$, then the energy $E = \int_{-\infty}^{\infty} \left(u_t^2 + c^2u_x^2\right)\, dx$ is constant.

Can you find a corresponding conserved quantity $\hat{E}$ for solutions of the equation $u_{tt} - c^2u_{xx} - bu = 0$?

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FOR STUDENTS ENROLLED IN 92.545.

Problem 6. (20 points)

(a) (15 points) Solve the equation $uu_x + u_y = 1$ with the initial condition $u(x, x) = 0$.
Hints: When finding the characteristic curves, solve for u before solving for x. When expressing t in terms of x and y, remember that $t = 0$ on the initial curve.

(b) (5 points) Find the domain of the solution $u(x, y)$ you found in part a.

Problem 7. (10 points)

On Homework Assignment # 3 you showed that if u is a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ on $-\infty < x < \infty, t > 0$ for which $u \to 0, u_x \to 0$, and $u_t \to 0$ as $x \to \pm\infty$, then the energy $E = \int_{-\infty}^{\infty} \left(u_t^2 + c^2 u_x^2\right)\, dx$ is constant.

Show that if u is a solution of the equation $u_{tt} - c^2 u_{xx} + au_t = 0$ $(a > 0)$ on $-\infty < x < \infty, t > 0$ for which $u \to 0, u_x \to 0$, and $u_t \to 0$ as $x \to \pm\infty$, then $\dfrac{dE}{dt} \leq 0$.

Note: The term au_t in the pde is a *dissipative* term that causes energy loss.