

92.445/545 Partial Differential Equations Spring 2013
Homework Assignment # 2 Solutions

1. Consider the pde $u_t + (4u^3 + 4u)u_x = 0$, and consider a weak solution in the form of a shock wave: $u = \begin{cases} u^- & \text{if } x < \gamma(t) \\ u^+ & \text{if } x > \gamma(t) \end{cases}$

Find the speed of propagation $\gamma'(t)$ of the shock wave.

$(4u^3 + 4u)u_x = \frac{\partial}{\partial x} [u^4 + 2u^2]$. Therefore, the flux term in this pde is given by $F(u) = u^4 + 2u^2$.

The Rankine-Hugoniot condition tells us that

$$\begin{aligned} \gamma'(t) &= \frac{F(u^+) - F(u^-)}{u^+ - u^-} \\ &= \frac{[(u^+)^4 + 2(u^+)^2] - [(u^-)^4 + 2(u^-)^2]}{u^+ - u^-} \\ &= \frac{[(u^+)^4 - (u^-)^4] + 2[(u^+)^2 - (u^-)^2]}{u^+ - u^-} \\ &= \frac{[(u^+)^2 + (u^-)^2][(u^+)^2 - (u^-)^2] + 2[(u^+)^2 - (u^-)^2]}{u^+ - u^-} \\ &= \frac{[(u^+)^2 - (u^-)^2] \{ [(u^+)^2 + (u^-)^2] + 2 \}}{u^+ - u^-} \\ &= \frac{[(u^+) - (u^-)][(u^+) + (u^-)] \{ [(u^+)^2 + (u^-)^2] + 2 \}}{u^+ - u^-} \\ &\Rightarrow \boxed{\gamma'(t) = [(u^+) + (u^-)] [(u^+)^2 + (u^-)^2 + 2]} \end{aligned}$$

2. Use D'Alembert's Formula to solve the following Cauchy problem.

$$\begin{aligned} u_{tt} - 9u_{xx} &= 0 && \text{on } -\infty < x < \infty, t > 0 \\ u(x, 0) &= 0 \\ u_t(x, 0) &= 2xe^{-x^2} \end{aligned}$$

D'Alembert's Formula says that the solution of the Cauchy problem

$$\begin{aligned} u_{tt} - c^2u_{xx} &= 0 && \text{on } -\infty < x < \infty, t > 0 \\ u(x, 0) &= f(x) \\ u_t(x, 0) &= g(x) \end{aligned}$$

is given by $u(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$. (See equation 4.11 on page 79 of the textbook.) In this problem, $c = 3$, $f(x) = 0$, and $g(x) = 2xe^{-x^2}$. Therefore,

$$u(x, t) = \frac{1}{2} [0 + 0] + \frac{1}{2 \cdot 3} \int_{x-3t}^{x+3t} 2se^{-s^2} ds = \frac{1}{6} \left[-e^{-s^2} \Big|_{x-3t}^{x+3t} \right] \Rightarrow \boxed{u(x, t) = \frac{1}{6} [e^{-(x-3t)^2} - e^{-(x+3t)^2}]}$$