92.445/545 Partial Differential Equations   Spring 2013
Homework Assignment # 3 Solutions

1. Solve the following Cauchy problem on a semi-infinite domain.

$$\begin{aligned} u_{tt} - 4u_{xx} &= 0 \qquad \text{on } 0 < x < \infty, t > 0 \\ u(x,0) &= xe^{-x} \\ u_t(x,0) &= 0 \\ u(0,t) &= 0 \end{aligned}$$

As we discussed in class, the solution to the Cauchy problem

$$\begin{aligned} u_{tt} - c^2u_{xx} &= 0 \qquad \text{on } 0 < x < \infty, t > 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \\ u(0,t) &= 0 \end{aligned}$$

equals the solution of the Cauchy problem

$$\begin{aligned} u_{tt} - c^2u_{xx} &= 0 \qquad \text{on } -\infty < x < \infty, t > 0 \\ u(x,0) &= \hat{f}(x) \\ u_t(x,0) &= \hat{g}(x) \end{aligned}$$

restricted to the interval $0 < x < \infty$. Here $\hat{f}$ and $\hat{g}$ are the functions obtained by extending f and g, respectively, to the entire real line as odd functions. In this problem, $f(x) = xe^{-x}$ and $g(x) = 0$. Their extensions as odd functions are $\hat{f}(x) = xe^{-|x|}$ and $\hat{g}(x) = 0$. In this problem, $c = 2$, so D'Alembert's Formula gives us

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds \\ &= \frac{1}{2}\left[(x+2t)e^{-|x+2t|} + (x-2t)e^{-|x-2t|}\right] + \frac{1}{2\cdot 2}\int_{x-2t}^{x+2t} 0\, ds \end{aligned}$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\left[(x+2t)e^{-|x+2t|} + (x-2t)e^{-|x-2t|}\right]}$$

2. Solve the following Cauchy problem on a semi-infinite domain.

$$\begin{aligned} u_{tt} - 9u_{xx} &= 0 \qquad \text{on } 0 < x < \infty, t > 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= \cos(x) \\ u_x(0,t) &= 0 \end{aligned}$$

As we discussed in class, the solution to the Cauchy problem

$$\begin{aligned} u_{tt} - c^2u_{xx} &= 0 \qquad \text{on } 0 < x < \infty, t > 0 \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \\ u_x(0,t) &= 0 \end{aligned}$$

equals the solution of the Cauchy problem

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= 0 \qquad \text{on } -\infty < x < \infty, t > 0 \\
u(x,0) &= \hat{f}(x) \\
u_t(x,0) &= \hat{g}(x)
\end{aligned}
$$

restricted to the interval $0 < x < \infty$. Here $\hat{f}$ and $\hat{g}$ are the functions obtained by extending f and g, respectively, to the entire real line as even functions. In this problem, $f(x) = 0$ and $g(x) = \cos(x)$. Their extensions as odd functions are $\hat{f}(x) = 0$ and $\hat{g}(x) = \cos(x)$. In this problem, $c = 3$, so D'Alembert's Formula gives us

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\ ds \\
&= \frac{1}{2}\left[0+0\right] + \frac{1}{2\cdot 3}\int_{x-3t}^{x+3t} \cos(s)\ ds \\
&= \frac{1}{6}\left[\sin(s)|_{x-3t}^{x+3t}\right]
\end{aligned}
$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{6}\left[\sin(x+3t) - \sin(x-3t)\right]}$$

FOR STUDENTS ENROLLED IN 92.545.

3. Suppose u is a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ on $-\infty < x < \infty, t > 0$ and suppose that $u \to 0, u_x \to 0$, and $u_t \to 0$ as $x \to \pm\infty$. Show that $E = \displaystyle\int_{-\infty}^{\infty} \left(u_t^2 + c^2 u_x^2\right)\ dx$ is constant.

Hints: Use the wave equation and look at $\dfrac{\partial}{\partial x}(u_t u_x)$

$$
\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\int_{-\infty}^{\infty}\left(u_t^2 + c^2 u_x^2\right)\ dx \\
&= \int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left(u_t^2 + c^2 u_x^2\right)\ dx \text{ using Leibniz's Rule} \\
&= \int_{-\infty}^{\infty}\left(2u_t u_{tt} + 2c^2 u_x u_{xt}\right)\ dx \\
&= 2\int_{-\infty}^{\infty}\left[u_t\left(c^2 u_{xx}\right) + c^2 u_x u_{xt}\right]\ dx
\end{aligned}
$$

using wave eqn.

$$
\begin{aligned}
&= 2c^2\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left[u_t u_x\right]\ dx \\
&= 2c^2\ u_t u_x|_{x\to-\infty}^{x\to\infty} \\
&= 0 \text{ because } u_x \to 0 \text{ and } u_t \to 0 \text{ as } x \to \pm\infty
\end{aligned}
$$

Since $\dfrac{dE}{dt} = 0$, E is a constant.