92.445/545 Partial Differential Equations   Spring 2013

Homework Assignment # 4 Solutions

1. (Pinchover and Rubinstein problem 3.1) Find the canonical form of the following pde. Be sure to show the change of coordinates that reduces the pde to canonical form.

$$u_{xx} - 6u_{xy} + 9u_{yy} = xy^2$$

Recall that the second order linear pde in 2 independent variables $au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$ is hyperbolic if $b^2 - ac > 0$, parabolic if $b^2 - ac = 0$, and elliptic if $b^2 - ac < 0$. Here $a = 1$, $2b = -6$, and $c = 9$, so $b^2 - ac = (-3)^2 - 1(9) = 0$. Therefore, this pde is parabolic in the entire xy plane.

As explained on page 69 of the textbook, the canonical variable η satisfies the equation $a\eta_x + b\eta_y = 0$ and ξ can be chosen to be any function which makes the Jacobian $\xi_x\eta_y - \xi_y\eta_x$ nonzero. In this problem, $a = 1$ and $b = -3$, so η must satisfy the equation $\eta_x - 3\eta_y = 0$. Solving this equation by the method of characteristics, we find that $\eta = f(3x + y)$. For simplicity, we take $\eta = 3x + y$. If we just take $\xi = x$, the Jacobian of the transformation becomes $\xi_x\eta_y - \xi_y\eta_x = (1)(1) - (0)(3) = 1 \neq 0$. We can therefore take $\boxed{\xi = x}$ and $\boxed{\eta = 3x + y}$. With this choice, we obtain

$$\begin{aligned}
u_x &= w_\xi\xi_x + w_\eta\eta_x = w_\xi + 3w_\eta \\
u_y &= w_\xi\xi_y + w_\eta\eta_y = 0 \cdot w_\xi + w_\eta = w_\eta \\
u_{xx} &= [w_{\xi\xi}\xi_x + w_{\xi\eta}\eta_x] + 3\,[w_{\eta\xi}\xi_x + w_{\eta\eta}\eta_x] = w_{\xi\xi} + 6w_{\xi\eta} + 9w_{\eta\eta} \\
u_{xy} &= [w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y] + 3\,[w_{\eta\xi}\xi_y + w_{\eta\eta}\eta_y] = w_{\xi\eta} + 3w_{\eta\eta} \\
u_{yy} &= [w_{\eta\xi}\xi_y + w_{\eta\eta}\eta_y] = w_{\eta\eta}
\end{aligned}$$

Therefore, the given pde $u_{xx} - 6u_{xy} + 9u_{yy} = xy^2$ becomes
$[w_{\xi\xi} + 6w_{\xi\eta} + 9w_{\eta\eta}] - 6\,[w_{\xi\eta} + 3w_{\eta\eta}] + 9\,[w_{\eta\eta}] = \xi\,(\eta - 3\xi)^2$, or $\boxed{w_{\xi\xi} = \xi\,(\eta - 3\xi)^2}$

You might get a different right-hand side if you chose a different transformation.

2. (Pinchover and Rubinstein problem 5.1) Solve the following IBVP.

$$\begin{aligned}
u_t &= 17u_{xx} \qquad \text{on } 0 < x < \pi, t > 0 \\
u(0,t) &= 0 \quad t \geq 0 \\
u(\pi,t) &= 0 \quad t \geq 0 \\
u(x,0) &= \begin{cases} 0 & 0 \leq x \leq \pi/2 \\ 2 & \pi/2 < x \leq \pi \end{cases}
\end{aligned}$$

As discussed in class and as shown on page 104 of Pinchover and Rubinstein, the solution of the heat equation with thermal diffusivity k over the interval $0 < x < L$ with $u(0,t) = 0$, $u(L,t) = 0$, and $u(x,0) = f(x)$ can be written as

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\pi^2 n^2 t/L^2}$$

where

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

In this problem $k = 17$ and $L = \pi$, so we have

$$B_n = \frac{2}{\pi}\int_0^\pi f(x)\sin\left(\frac{n\pi x}{\pi}\right)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} 0\cdot\sin(nx)\,dx + \int_{\pi/2}^{\pi} 2\cdot\sin(nx)\,dx\right] = -\frac{4}{n\pi}\left[\cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right]$$

Because $\cos(n\pi) = (-1)^n$, this gives us $\boxed{u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right]\sin(nx)e^{-17n^2t}}$

3. (Pinchover and Rubinstein problem 5.4) Solve the following IBVP. Hint: Use a trig identity to replace $\sin^3(x)$ by an equivalent expression.

$$\begin{aligned} u_{tt} &= u_{xx} && \text{on } 0 < x < \pi, t > 0 \\ u(0,t) &= 0 && t \geq 0 \\ u(\pi,t) &= 0 && t \geq 0 \\ u(x,0) &= \sin^3(x) && 0 \leq x \leq \pi \\ u_t(x,0) &= \sin(2x) && 0 \leq x \leq \pi \end{aligned}$$

As discussed in class, the solution of the wave equation with wave speed c over the interval $0 < x < L$ with $u(0,t) = 0$, $u(L,t) = 0$, $u(x,0) = f(x)$, and $u_t(x,0) = g(x)$ can be written as

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{\pi nct}{L}\right) + B_n\sin\left(\frac{\pi nct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

where

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

are the coefficients in the Fourier sine series expansion of f and

$$B_n = \frac{2}{\pi nc}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

are $\frac{L}{\pi nc}$ times the Fourier sine series coefficients of g.

$\sin^3(x) = \sin(x)\sin^2(x) = \sin(x)\left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right] = \frac{1}{2}\sin(x) - \frac{1}{2}\sin(x)\cos(2x)$
$= \frac{1}{2}\sin(x) - \frac{1}{2}\left\{\frac{1}{2}\left[\sin(x+2x) + \sin(x-2x)\right]\right\} = \frac{3}{4}\sin(x) - \frac{1}{4}\sin(3x)$

Therefore, the Fourier sine series for $f(x) = \sin^3(x)$ has only 2 nonzero terms: $A_1 = \frac{3}{4}$ and $A_3 = -\frac{1}{4}$.

The Fourier sine series for $g(x) = \sin(2x)$ has only one nonzero term, with coefficient equal to 1, corresponding to $n = 2$.

In this problem, $c = 1$ and $L = \pi$, so we have $B_2 = \dfrac{\pi}{\pi(2)(1)} \cdot 1 = \dfrac{1}{2}$

Therefore,

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{\pi nct}{L}\right) + B_n \sin\left(\frac{\pi nct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty} \left[A_n \cos(nt) + B_n \sin(nt)\right] \sin(nx) \\ &= A_1 \cos(t)\sin(x) + B_2 \sin(2t)\sin(2x) + A_3 \cos(3t)\sin(3x) \end{aligned}$$

so $\boxed{u(x,t) = \frac{3}{4}\cos(t)\sin(x) + \frac{1}{2}\sin(2t)\sin(2x) - \frac{1}{4}\cos(3t)\sin(3x)}$

FOR STUDENTS ENROLLED IN 92.545.

4. (Pinchover and Rubinstein problem 3.12) Consider the pde $u_{xx} + yu_{yy} = 0$.

a. Find the domain on which the given pde is elliptic.

Recall that the second order linear pde in 2 independent variables $au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$ is hyperbolic if $b^2 - ac > 0$, parabolic if $b^2 - ac = 0$, and elliptic if $b^2 - ac < 0$. Here $a = 1$, $2b = 0$, and $c = y$, so $b^2 - ac = 0^2 - 1(y) = -y$. Therefore, this pde is elliptic in the upper half-plane $y > 0$.

b. Find the canonical form of the given pde on the domain you found in part a.

We will use the change of variables given by $\xi = \Re(\phi)$ and $\eta = \Im(\phi)$, where ϕ is a solution of

$$a\phi_x + \left(b + i\sqrt{ac - b^2}\right)\phi_y = 0$$

Here $a = 1$, $b = 0$, and $c = y$, so we need a solution of $\phi_x + i\sqrt{y}\phi_y = 0$. Suppose the initial data are given on the positive y axis, so the initial data curve Γ is parameterized by $x = 0,\ y = s$. The characteristic curves satisfy the equations $\dfrac{dx}{dt} = 1$, $\dfrac{dy}{dt} = i\sqrt{y}$. $\dfrac{dx}{dt} = 1$, $x = 0$ on $t = 0 \Rightarrow x = t$. $\dfrac{dy}{dt} = i\sqrt{y}$, $y = s$ on $t = 0 \Rightarrow y^{-1/2}\,dy = i\,dt \Rightarrow 2y^{1/2} = it + 2s^{1/2}$. The pde for ϕ tells us that $\dfrac{d\phi}{dt} = 0$ on characteristics, so $\phi = f(s)$. For convenience we take $f(s) = 2s^{1/2}$, so $\phi = 2s^{1/2} = 2y^{1/2} - it = 2y^{1/2} - ix$. Then $\boxed{\xi = \Re(\phi) = 2y^{1/2}}$ and $\boxed{\eta = \Im(\phi) = -x}$. With this choice, we obtain

$$\begin{aligned} u_x &= w_\xi \xi_x + w_\eta \eta_x = -w_\eta \\ u_y &= w_\xi \xi_y + w_\eta \eta_y = y^{-1/2} w_\xi \\ u_{xx} &= -\left[w_{\eta\xi}\xi_x + w_{\eta\eta}\eta_x\right] = w_{\eta\eta} \\ u_{yy} &= -\frac{1}{2}y^{-3/2}w_\xi + y^{-1/2}\left[w_{\xi\xi}\xi_y + w_{\xi\eta}\eta_y\right] = -\frac{1}{2}y^{-3/2}w_\xi + y^{-1}w_{\xi\xi} \end{aligned}$$

Therefore, the given pde $u_{xx} + yu_{yy} = 0$ becomes $w_{\eta\eta} + y\left[-\frac{1}{2}y^{-3/2}w_\xi + y^{-1}w_{\xi\xi}\right] = 0$, or

$$\boxed{w_{\xi\xi} + w_{\eta\eta} - \frac{1}{\xi}w_\xi = 0}$$