92.445/545 Partial Differential Equations Spring 2013

Homework Assignment # 5 Solutions

PLEASE SHOW ALL WORK! You will not receive full credit if you do not show your work. You may work together, but everyone must turn in his/her own homework set.

1. Solve the following IBVP for the nonhomogeneous wave equation.

$$\begin{aligned} u_{tt} - 4u_{xx} &= 6\sin(t)\sin(x) \qquad \text{on } 0 < x < \pi, t > 0 \\ u(0,t) &= 0 \quad t \geq 0 \\ u(\pi,t) &= 0 \quad t \geq 0 \\ u(x,0) &= 0 \quad 0 \leq x \leq \pi \\ u_t(x,0) &= 0 \quad 0 \leq x \leq \pi \end{aligned}$$

In view of the boundary conditions $u(0,t) = 0$ and $u(\pi,t) = 0$, we look for a solution in the form $u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(nx)$.

$$\begin{aligned} u_{tt} - 4u_{xx} &= 6\sin(t)\sin(x) \Rightarrow \\ \sum_{n=1}^{\infty}\left[T_n''(t) + 4n^2T_n(t)\right]\sin(nx) &= 6\sin(t)\sin(x) \Rightarrow \end{aligned}$$

$$\begin{cases} T_1''(t) + 4T_1(t) = 6\sin(t) \\ T_n''(t) + 4n^2T_n(t) = 0 \quad (n > 1) \end{cases}$$

The characteristic equation for the ode $T_n''(t) + 4n^2T_n(t) = 0$ is $r^2 + 4n^2 = 0$, the roots of which are $r = \pm 2ni$. Therefore, $T_n(t) = A_n\cos(2nt) + B_n\sin(2nt)$ for $n > 1$.

The solution of the ode $T_1''(t) + 4T_1(t) = 6\sin(t)$ is the sum of the complementary solution T_c and a particular solution T_p. The complementary solution is the general solution of $T_1''(t) + 4T_1(t) = 0$. The characteristic equation is $r^2 + 4 = 0$ which has roots $r = \pm 2i$, so $T_c = A_1\cos(2t) + B_1\sin(2t)$.

Since the nonhomogeneous term in the ode $T_1''(t) + 4T_1(t) = 6\sin(t)$ is $6\sin(t)$, we guess that a particular solution has the form $T_p = A\sin(t) + B\cos(t)$.

$T = A\sin(t) + B\cos(t) \Rightarrow T' = A\cos(t) - B\sin(t) \Rightarrow T'' = -A\sin(t) - B\cos(t) \Rightarrow$
$T'' + 4T = -A\sin(t) - B\cos(t) + 4[A\sin(t) + B\cos(t)] = 3A\sin(t) + 3B\cos(t)$.
We want this to equal the nonhomogeneous term $6\sin(t)$, so we choose $A = 2$ and $B = 0$.

Therefore, $T_p = 2\sin(t)$ and $T_1(t) = T_c + T_p = A_1\cos(2t) + B_1\sin(2t) + 2\sin(t)$.

It follows that $u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin(nx) = T_1(t)\sin(x) + \sum_{n=2}^{\infty} T_n(t)\sin(nx) =$

$[A_1\cos(2t) + B_1\sin(2t) + 2\sin(t)]\sin(x) + \sum_{n=2}^{\infty}[A_n\cos(2nt) + B_n\sin(2nt)]\sin(nx) =$

$2\sin(t)\sin(x) + \sum_{n=1}^{\infty}[A_n\cos(2nt) + B_n\sin(2nt)]\sin(nx)$

$u(x,0) = 0 \Rightarrow 2\sin(0)\sin(x) + \sum_{n=1}^{\infty}[A_n\cos(0) + B_n\sin(0)]\sin(nx) = 0 \Rightarrow \sum_{n=1}^{\infty} A_n\sin(nx) = 0 \Rightarrow A_n = 0 \ (n \geq 1)$

$$u(x,t) = 2\sin(t)\sin(x) + \sum_{n=1}^{\infty}[A_n\cos(2nt) + B_n\sin(2nt)]\sin(nx) \Rightarrow$$

$$u_t(x,t) = 2\cos(t)\sin(x) + \sum_{n=1}^{\infty}[-2nA_n\sin(2nt) + 2nB_n\cos(2nt)]\sin(nx)$$

$$u_t(x,0) = 0 \Rightarrow 2\cos(0)\sin(x) + \sum_{n=1}^{\infty}[-2nA_n\sin(0) + 2nB_n\cos(0)]\sin(nx) = 0 \Rightarrow$$

$$2\sin(x) + \sum_{n=1}^{\infty}2nB_n\sin(nx) = 0 \Rightarrow 2 + 2B_1 = 0 \text{ and } 2nB_n = 0\ (n \geq 2) \Rightarrow B_1 = -1 \text{ and } B_n = 0\ (n \geq 2)$$

Therefore, $\boxed{u(x,t) = 2\sin(t)\sin(x) - \sin(2t)\sin(x) = [2\sin(t) - \sin(2t)]\sin(x)}$

2. Solve the following IBVP for the heat equation with nonzero boundary conditions.

$$\begin{aligned} u_t &= u_{xx} \qquad \text{on } 0 < x < \pi, t > 0 \\ u(0,t) &= 2 \quad t \geq 0 \\ u(\pi,t) &= 2 \quad t \geq 0 \\ u(x,0) &= 2 + \sin(2x) \quad 0 \leq x \leq \pi \end{aligned}$$

As discussed in class, we look for a solution in the form $u(x,t) = u_1(x) + u_2(x,t)$ where $u_1(x)$ is a linear function satisfying the boundary conditions and $u_2(x,t)$ is a solution of the IBVP

$$\begin{aligned} u_{2t} &= u_{2xx} \qquad \text{on } 0 < x < \pi, t > 0 \\ u(0,t) &= 0 \quad t \geq 0 \\ u(\pi,t) &= 0 \quad t \geq 0 \\ u(x,0) &= 2 + \sin(2x) - u_1(x) \quad 0 \leq x \leq \pi \end{aligned}$$

In this problem, because the boundary conditions are $u(0,t) = 2$ and $u(\pi,t) = 2$, $u_1(x)$ is just the constant function $u_1(x) = 2$. Therefore, u_2 is a solution of the IBVP

$$\begin{aligned} u_{2t} &= u_{2xx} \qquad \text{on } 0 < x < \pi, t > 0 \\ u(0,t) &= 0 \quad t \geq 0 \\ u(\pi,t) &= 0 \quad t \geq 0 \\ u(x,0) &= \sin(2x) \quad 0 \leq x \leq \pi \end{aligned}$$

As discussed in class and as described on pages 99 - 104 of Pinchover and Rubinstein, the solution of this IBVP can be expressed in the form $u_2(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-kn^2\pi^2 t/L^2} = \sum_{n=1}^{\infty} B_n \sin(nx)\, e^{-n^2 t}$.

$$u_2(x,0) = \sin(2x) \Rightarrow \sum_{n=1}^{\infty} B_n \sin(nx)\, e^0 = \sin(2x) \Rightarrow B_2 = 1,\ B_n = 0\ (n \neq 2) \Rightarrow \boxed{u(x,t) = 2 + \sin(2x)e^{-4t}}$$