92.445/545 Partial Differential Equations Spring 2013

Homework Assignment # 6 Solutions

1. Solve the following BVP for the Laplace equation.

$$\begin{aligned} \Delta u &= 0 && \text{on } 0 < x < \pi, 0 < y < \pi \\ u(x,0) &= \sin(2x) && 0 < x < \pi \\ u(x,\pi) &= 0 && 0 < x < \pi \\ u(0,y) &= \sin(y) && 0 < y < \pi \\ u(\pi,y) &= 0 && 0 < y < \pi \end{aligned}$$

As stated on the handout on solving the Laplace equation via separation of variables, the solution of the boundary value problem

$$\begin{aligned} \Delta u &= 0 \text{ on } 0 < x < b,\ 0 < y < d \\ u(0,y) &= f(y)\ 0 < y < d \\ u(b,y) &= g(y)\ 0 < y < d \\ u(x,0) &= h(x)\ 0 < x < b \\ u(x,d) &= k(x)\ 0 < x < b \end{aligned}$$

can be expressed as

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty} \left\{ \sin\left(\frac{n\pi y}{d}\right) \left[A_n \sinh\left(\frac{n\pi x}{d}\right) + B_n \sinh\left(\frac{n\pi(x-b)}{d}\right) \right] \right. \\ &\quad \left. + \sin\left(\frac{n\pi x}{b}\right) \left[C_n \sinh\left(\frac{n\pi y}{b}\right) + D_n \sinh\left(\frac{n\pi(y-d)}{b}\right) \right] \right\} \end{aligned}$$

where $A_n \sinh\left(\frac{n\pi b}{d}\right)$ equals the nth Fourier sine series coefficient of $g(y)$ on $0 < y < d$, $-B_n \sinh\left(\frac{n\pi b}{d}\right)$ equals the nth Fourier sine series coefficient of $f(y)$ on $0 < y < d$, $C_n \sinh\left(\frac{n\pi d}{b}\right)$ equals the nth Fourier sine series coefficient of $k(x)$ on $0 < x < b$, and $-D_n \sinh\left(\frac{n\pi d}{b}\right)$ equals the nth Fourier sine series coefficient of $h(x)$ on $0 < x < b$.

In this problem, $b = d = \pi$, so

$$u(x,y) = \sum_{n=1}^{\infty} \left\{ \sin(ny) \left[A_n \sinh(nx) + B_n \sinh(n(x-\pi)) \right] + \sin(nx) \left[C_n \sinh(ny) + D_n \sinh(n(y-\pi)) \right] \right\}$$

$u(x,0) = \sin(2x) \Rightarrow$

$$\begin{aligned} \sin(2x) &= \sum_{n=1}^{\infty} \left\{ \sin(0) \left[A_n \sinh(nx) + B_n \sinh(n(x-\pi)) \right] + \sin(nx) \left[C_n \sinh(0) + D_n \sinh(n(0-\pi)) \right] \right\} \\ &= \sum_{n=1}^{\infty} \sin(nx) \left[-D_n \sinh(n\pi) \right] \end{aligned}$$

$\Rightarrow D_n = 0$ for $n \neq 2$, $D_2 = -1/\sinh(2\pi)$

$u(x,\pi) = 0 \Rightarrow$

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} \{\sin(n\pi)\,[A_n \sinh(nx) + B_n \sinh(n(x-\pi))] \;+ \sin(nx)\,[C_n \sinh(n\pi) + D_n \sinh(n(\pi-\pi))]\} \\ &= \sum_{n=1}^{\infty} \sin(nx)\,[C_n \sinh(n\pi)] \end{aligned}$$

$\Rightarrow C_n = 0$ for all n

$u(0,y) = \sin(y) \Rightarrow$

$$\begin{aligned} \sin(y) &= \sum_{n=1}^{\infty} \{\sin(ny)\,[A_n \sinh(0) + B_n \sinh(n(0-\pi))] \;+ \sin(0)\,[C_n \sinh(ny) + D_n \sinh(n(y-\pi))]\} \\ &= \sum_{n=1}^{\infty} \sin(ny)\,[-B_n \sinh(n\pi))] \end{aligned}$$

$\Rightarrow B_n = 0$ for $n > 1$, $B_1 = -1/\sinh(\pi)$

$u(\pi,y) = 0 \Rightarrow$

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} \{\sin(ny)\,[A_n \sinh(n\pi) + B_n \sinh(n(\pi-\pi))] \;+ \sin(n\pi)\,[C_n \sinh(ny) + D_n \sinh(n(y-\pi))]\} \\ &= \sum_{n=1}^{\infty} \sin(ny)\,[A_n \sinh(n\pi))] \end{aligned}$$

$\Rightarrow A_n = 0$ for all n

Therefore, $u(x,y) = -\dfrac{1}{\sinh(\pi)} \sin(y)\sinh(x-\pi) - \dfrac{1}{\sinh(2\pi)} \sin(2x)\sinh(2(y-\pi)) \Rightarrow$

$$\boxed{u(x,y) = \frac{1}{\sinh(\pi)} \sin(y)\sinh(\pi - x) + \frac{1}{\sinh(2\pi)} \sin(2x)\sinh(2(\pi-y))}$$

2. (Pinchover & Rubinstein, problem 7.7b) Find a function u that is harmonic on the disk $x^2 + y^2 < 6$ and that satisfies the boundary condition $u(x,y) = y + y^2$ on the disk's boundary. Write your answer in terms of Cartesian coordinates.

As discussed in class and on the handout on solving the Laplace equation via separation of variables, the solution for the Laplace equation on the interior of a disk can be written in the form

$$w(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \,[A_n \cos(n\theta) + B_n \sin(n\theta)]$$

On the boundary $r = \sqrt{6}$ we have $w = y + y^2 = \sqrt{6}\sin(\theta) + \left(\sqrt{6}\right)^2 \sin^2(\theta)$

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3. (Pinchover & Rubinstein, problem 7.3) Solve the following BVP for the reduced Helmholtz equation with $k = 1$. Hint: Look for a solution in the form $u(x,y) = \sum_{n=1}^{\infty} X_n(x)\sin(ny)$

$$\begin{aligned} \Delta u - u &= 0 \qquad \text{on } 0 < x < \pi, 0 < y < \pi \\ u(0,y) &= 1 \quad 0 < y < \pi \\ u(\pi,y) &= 0 \quad 0 < y < \pi \\ u(x,0) &= 0 \quad 0 < x < \pi \\ u(x,\pi) &= 0 \quad 0 < x < \pi \end{aligned}$$

$$u(x,y) = \sum_{n=1}^{\infty} X_n(x)\sin(ny) \Rightarrow u_{xx} = \sum_{n=1}^{\infty} X_n''(x)\sin(ny) \text{ and } u_{yy} = \sum_{n=1}^{\infty} -n^2 X_n(x)\sin(ny)$$

Therefore, $\Delta u - u = 0 \Rightarrow u_{xx} + u_{xx} - u = 0 \Rightarrow \sum_{n=1}^{\infty}\left[X_n''(x)\sin(ny) - n^2 X_n(x)\sin(ny) - X_n(x)\sin(ny)\right] = 0$

$\Rightarrow \sum_{n=1}^{\infty} \sin(ny)\left[X_n''(x) - \left(n^2+1\right)X_n(x)\right] = 0 \Rightarrow X_n''(x) - \left(n^2+1\right)X_n(x) = 0,\ \ n = 1, 2, 3, \ldots$

The characteristic equation for the linear constant-coefficient ode $X_n''(x) - \left(n^2+1\right)X_n(x) = 0$ is $r^2 - (n^2+1) = 0$, the roots of which are $r = \pm\sqrt{n^2+1}$.

Therefore, $X_n(x) = A_n \sinh\left(\left(\sqrt{n^2+1}\right)x\right) + B_n \sinh\left(\left(\sqrt{n^2+1}\right)(x-\pi)\right)$ and

$u(x,y) = \sum_{n=1}^{\infty} \sin(ny)\left[A_n \sinh\left(\left(\sqrt{n^2+1}\right)x\right) + B_n \sinh\left(\left(\sqrt{n^2+1}\right)(x-\pi)\right)\right]$

$u(\pi,y) = 0 \Rightarrow$

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} \sin(ny)\left[A_n \sinh\left(\left(\sqrt{n^2+1}\right)\pi\right) + B_n \sinh\left(\left(\sqrt{n^2+1}\right)(\pi-\pi)\right)\right] \\ &= \sum_{n=1}^{\infty} \sin(ny)\left[A_n \sinh\left(\left(\sqrt{n^2+1}\right)\pi\right)\right] \end{aligned}$$

$\Rightarrow A_n \sinh\left(\left(\sqrt{n^2+1}\right)\pi\right) = 0$ for all $n \Rightarrow A_n = 0$ for all n.

$u(0,y) = 1 \Rightarrow$

$$\begin{aligned} 1 &= \sum_{n=1}^{\infty} \sin(ny)\left[A_n \sinh\left(\left(\sqrt{n^2+1}\right)0\right) + B_n \sinh\left(\left(\sqrt{n^2+1}\right)(0-\pi)\right)\right] \\ &= \sum_{n=1}^{\infty} \sin(ny)\left[B_n \sinh\left(-\left(\sqrt{n^2+1}\right)\pi\right)\right] \end{aligned}$$

$\Rightarrow -B_n \sinh\left(\left(\sqrt{n^2+1}\right)\pi\right)$ equals the nth Fourier sine series coefficient for the constant function 1 on $0 < y < \pi$:

$$
\begin{aligned}
-B_n \sinh\left(\left(\sqrt{n^2+1}\right)\pi\right) &= \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin(ny)\ dy \\
&= -\frac{2}{n\pi}\cos(ny)\Big|_0^{\pi} \\
&= -\frac{2}{n\pi}\left[\cos(n\pi) - cos(0)\right] \\
&= \begin{cases} 4/(n\pi) & n \text{ odd} \\ 0 & n \text{ even} \end{cases}
\end{aligned}
$$

$$
\Rightarrow B_n = \begin{cases} -4/\left(n\pi \sinh\left(\left(\sqrt{n^2+1}\right)\pi\right)\right) & n \text{ odd} \\ 0 & n \text{ even} \end{cases}
$$

Therefore, $\boxed{u(x,y) = \frac{4}{\pi}}$