92.445/545 Partial Differential Equations Spring 2013

Optional Homework Assignment # 7 Solutions

1. a) Find the Fourier series for the sawtooth function given by $f(x) = x$ on the interval $-\pi \le x \le \pi$.

The Fourier series for the function f on the interval $-L \le x \le L$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx$.

f is an odd function, so $a_n = 0$ for $n \ge 0$. $L = \pi$, so

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\, dx = \frac{2(-1)^{n+1}}{n}$$

Therefore, the Fourier series for f on the interval $-\pi \le x \le \pi$ is given by $\boxed{\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nx)}$

b) To what value does the Fourier series converge at $x = \pi/2$?

f is continuous at $\pi/2$, so the Fourier series converges to $\boxed{f(\pi/2) = \pi/2}$

c) To what value does the Fourier series converge at $x = \pi$?

At $\pm L$ the Fourier series converges to $\frac{1}{2}\left[f(-L) + f(L)\right] = \frac{1}{2}\left[-\pi + \pi\right] = \boxed{0}$.

2. Use the Fourier Transform to solve the following IVP for the heat equation:

$$u_t = u_{xx} \text{ on } -\infty < x < \infty,\ u(x,0) = e^{-x^2/2}$$

Hints: $\mathcal{F}\left\{e^{-ax^2}\right\} = \sqrt{\frac{\pi}{a}}e^{-k^2/(4a)}$ and $\mathcal{F}^{-1}\left\{e^{-bk^2}\right\} = \frac{1}{2\sqrt{\pi b}}e^{-x^2/(4b)}$

$u_t = u_{xx} \Rightarrow \mathcal{F}\{u_t\} = \mathcal{F}\{u_{xx}\} \Rightarrow U_t = -k^2 U,$ where $U = \mathcal{F}\{u\}$.

$u(x,0) = e^{-x^2/2} \Rightarrow \mathcal{F}\{u(x,0)\} = \mathcal{F}\left\{e^{-x^2/2}\right\} \Rightarrow U(k,0) = \sqrt{2\pi}e^{-k^2/2}$

$U_t(k,t) = -k^2 U(k,t),\ U(k,0) = \sqrt{2\pi}e^{-k^2/2} \Rightarrow U(k,t) = \left[\sqrt{2\pi}e^{-k^2/2}\right]e^{-k^2 t} = \sqrt{2\pi}e^{-k^2(t+1/2)} \Rightarrow$

$u(x,t) = \sqrt{2\pi}\left[\frac{1}{2\sqrt{\pi(t+1/2)}}e^{-x^2/(4(t+1/2))}\right] \Rightarrow \boxed{u(x,t) = \frac{1}{\sqrt{2t+1}}e^{-x^2/(4t+2)}}$