FIBRATIONS OF SIMPLICIAL SETS

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ABSTRACT. There are infinitely many variants of the notion of Kan fibration that, together with suitable choices of cofibrations and the usual notion of weak equivalence of simplicial sets, satisfy Quillen's axioms for a homotopy model category. The combinatorics underlying these fibrations is purely finitary and seems interesting both for its own sake and for its interaction with homotopy types. To show that these notions of fibration are indeed distinct, one needs to understand how iterates of Kan's Ex functor act on graphs and on nerves of small categories.

1. INTRODUCTION

The definition of fibration that now bears his name was introduced by Daniel Kan in 1957, and remains a cornerstone of simplicial algebraic topology. A decade later, Quillen axiomatized homotopy theory via his notion of a *model category* that comes equipped with three distinguished classes of morphisms: fibrations, weak equivalences and cofibrations. The category of simplicial sets, where Kan fibrations, topological (also called 'combinatorial') weak equivalences, and monomorphisms serve these roles, remains the primordial example of a homotopy model category. The goal of this article is to prove the following

Theorem: There exists a countably infinite properly increasing chain of subcategories of $SSet$

$$\mathsf{fib}_0 \subsetneqq \mathsf{fib}_1 \subsetneqq \mathsf{fib}_3 \subsetneqq \ldots \subsetneqq \mathsf{fib}_n \subsetneqq \ldots$$

and corresponding countable properly decreasing chain of subcategories

$$\mathsf{cof}_0 \supsetneqq \mathsf{cof}_1 \supsetneqq \mathsf{cof}_3 \supsetneqq \ldots \supsetneqq \mathsf{cof}_n \supsetneqq \ldots$$

such that for each n, fib_n together with cof_n and the usual (topological) notion of weak equivalence provide a Quillen model structure on $SSet$. Here cof_0 is the class of monomorphisms and fib_0 that of Kan fibrations.

This phenomenon of "variable (co)fibrations" is quite prevalent in Quillen model categories. Recall that Quillen in [15] already proves the existence of two different notions of cofibration on the category of chain complexes of modules (with one and the same definition of weak equivalence, namely quasi-isomorphisms); on the category of simplicial diagrams, with objectwise weak equivalences, one has the cofibrations of Bousfield–Kan [3] and Heller [7]; on cosimplicial spaces, i.e. cosimplicial diagrams of simplicial sets, there is yet another one due to Reedy; on the category of symmetric spectra, again at least three.

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There are at least two systematic methods of generating new cofibration classes within suitable model categories. If $\mathcal{M} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{N}$ is a Quillen adjunction between model categories $\mathcal{M} = \langle \mathsf{cof}_{\mathcal{M}}, \mathsf{W}_{\mathcal{M}}, \mathsf{fib}_{\mathcal{M}} \rangle$ and $\mathcal{N} = \langle \mathsf{cof}_{\mathcal{N}}, \mathsf{W}_{\mathcal{N}}, \mathsf{fib}_{\mathcal{N}} \rangle$ such that the right adjoint R preserves and reflects weak equivalences, and preserves fibrations, then under a wide range of conditions (see below) $\langle \mathrm{LLP}, \mathsf{W}_{\mathcal{M}}, R^{-1}(\mathsf{fib}_{\mathcal{N}}) \rangle$ will be a model structure on $\mathcal{M}$; here $R^{-1}(\mathsf{fib}_{\mathcal{N}})$ is the pre-image of the class $\mathsf{fib}_{\mathcal{N}}$ under R, and LLP is the class of morphisms with the left lifting property with respect to $\mathsf{W}_{\mathcal{M}} \cap R^{-1}(\mathsf{fib}_{\mathcal{N}})$. Note that $\mathsf{fib}_{\mathcal{M}} \subseteq R^{-1}(\mathsf{fib}_{\mathcal{N}})$ by assumption. In case $\mathcal{M} = \mathcal{N}$, the device can be iterated. This is what we will do for simplicial sets, with Kan's Ex functor playing the role of R.

The other method relies on a theorem of Jeff Smith, see Thm. 1.7 of [2]. Let us agree that from now on, we are only concerned with locally presentable model categories $\mathcal{M}$ whose category of weak equivalences W is accessible, and all cofibration classes will possess a *set* of generators. Suppose $\langle \mathsf{cof}_0, \mathsf{W}, \mathsf{fib}_0 \rangle$ and $\langle \mathsf{cof}_1, \mathsf{W}, \mathsf{fib}_1 \rangle$ are both model structures on M, and $\mathsf{cof}_0 \subsetneqq \mathsf{cof}_1$. (This situation is much less special than it might seem; the algebraic examples listed above, the sheaf-theoretic ones below, and many more are like this. See [2] for an elaboration.) Given some set of morphisms I, write $\mathsf{cof}(I)$ for the class of morphisms generated by pushouts, transfinite compositions and retracts from I (this being the left lifting class corresponding to the right lifting class of I), and write $\mathsf{fib}(I)$ for the class of morphisms with the right lifting property with respect to $\mathsf{W} \cap \mathsf{cof}(I)$. Let I be any set of morphisms such that $I \subset \mathsf{cof}_1$ and $\mathsf{cof}_0 \subseteq \mathsf{cof}(I)$. Then it follows from Jeff Smith's theorem that $\langle \mathsf{cof}(I), \mathsf{W}, \mathsf{fib}(I) \rangle$ is a model structure on $\mathcal{M}$ as well.

Note that $\mathsf{cof}_0 \subseteq \mathsf{cof}(I) \subseteq \mathsf{cof}_1$. If the inclusions are strict, we have managed to place an "intermediate" class of cofibrations between the cof_0 and cof_1 that were assumed to exist. It is easy to achieve $\mathsf{cof}_0 \subsetneqq \mathsf{cof}(I) \subseteq \mathsf{cof}_1$: find a set I_0 such that $\mathsf{cof}_0 = \mathsf{cof}(I_0)$, and let I be any set of morphisms with $I_0 \subset I \subset \mathsf{cof}_1$ such that I is not a subset of cof_0. What is much harder to establish is the strictness of the second inclusion. If I includes a set of generators for cof_1 then $\mathsf{cof}(I) = \mathsf{cof}_1$, of course. Proving that $\mathsf{cof}(I) \subsetneqq \mathsf{cof}_1$ seems to require showing that appropriate objects are *not* injective with respect to certain morphisms, a somewhat unusual problem. Nonetheless, *a priori* there is a proper class of choices for I, thus room for a proper class of intermediate cofibration classes.

Note that cofibration classes intermediate between the Bousfield–Kan class and all monomorphisms have already been constructed for (pre)sheaves of simplicial sets over a Grothendieck site, weak equivalences being the usual stalkwise ones. (See Beke [2] Example 2.17, Jardine [11], Isaksen [10].) Unlike the class of all monomorphisms, these intermediate cofibration classes are not functorial with respect to all geometric morphisms between toposes. But perhaps it is time to ask outright

Question 1.1. (a) Is there a proper class of distinct cofibration classes on $SSet$, where weak equivalences are taken to be the usual topological ones?
(b) Is there a maximal among these cofibration classes?
(c) If so, is the maximal one the class of all monomorphisms? (Equivalently, does every axiomatic class of fibrations include the Kan fibrations?)

One would venture that the behavior of $SSet$ is paradigmatic among all combinatorial model categories. As regards (a), note that in general there is no upper bound on the cardinality of possible cofibration classes for a model category (with fixed weak equivalences, hence the same homotopy category), nor do these classes have to be ordered linearly by inclusion. As regards (b), it is quite easy to see that for combinatorial model categories, suprema exist for *sets* of cofibration classes in the partial order they form under inclusion. This, and the ensuing “Quillen uniqueness” for cofibrations, is investigated in [1]. As regards (c), the left-determined model structures of Rosický and Tholen [16] may well be relevant.

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2. SUBDIVIDED COFIBRATIONS

In this paper, our concern is showing that $SSet$ contains an infinite properly decreasing chain of cofibration classes. Topologically, they are all equivalent; the variability is due to the combinatorics of simplices. A map belongs to the n^{th} exotic sense of fibration in Thm. 2.2, quite simply, if it becomes a Kan fibration after n iterations of Kan's simplicial extension functor Ex. Proving that the fib_n, thus defined, form part of a Quillen model structure on $SSet$ is straightforward. To prove the strict monotonicity of the inclusion $\mathsf{fin}_n \subsetneqq \mathsf{fib}_{n+1}$, it is enough to show that there is a simplicial set that becomes a Kan complex after exactly $n+1$ iterations of Ex. Using the small object argument, one can generate a fairly explicit family of simplicial sets X such that $\mathrm{Ex}^{n+1}(X)$ is fibrant (in the ordinary sense). The hard part is finding an X among them such that $\mathrm{Ex}^n(X)$ is not yet fibrant. We show, by an ad hoc path-length argument in the category of graphs, that the fibrantization (in the $n+1^{st}$ sense) of the n^{th} subdivision of the horn Λ^0_2 is such an X. Many aspects of the combinatorics of iterated Ex remain delightfully mysterious; some surprising connections will be pointed out in the closing section of this paper.

Let us recall what Ex is. Thinking of the n-simplex Δ_n (for the moment only) as a combinatorial simplicial complex, it has a barycentric subdivision $\mathrm{sd}\,\Delta_n$. Using the natural partial ordering of its vertices, $\mathrm{sd}\,\Delta_n$ can be made into a simplicial set. The correspondence $[n] \mapsto \mathrm{sd}\,\Delta_n$ in fact extends to a functor $\Delta \xrightarrow{\mathrm{sd}} SSet$, where Δ is the simplicial indexing category. Such a functor generates a self-adjunction of $SSet$ in a standard categorical manner

$$\begin{tikzcd} SSet \arrow[r, shift left, "\mathrm{Ex}"] & SSet \arrow[l, shift left, "\mathrm{Sd}"] \\ & \Delta \arrow[ul, "\mathrm{sd}"] \arrow[u, "y"'] \end{tikzcd}$$

Here $y : \Delta \to SSet$ is the Yoneda embedding, Sd is the left Kan extension of sd along y, and Ex is the right adjoint of Sd; for $X \in SSet$, the set of n-simplices of $\mathrm{Ex}(X)$ is given by $\hom_{SSet}(\mathrm{sd}\,\Delta_n, X)$, and the face and degeneracy maps of $\mathrm{Ex}(X)$ are defined via those between the $\mathrm{sd}\,\Delta_n$.

There exist morphisms (the “last vertex maps”) $\mathrm{sd}\,\Delta_n \to \Delta_n$ which induce a natural transformation (in fact, inclusion) $X \xrightarrow{\eta_X} \mathrm{Ex}(X)$ that is a weak equivalence for all X. From the naturality of

$$\begin{tikzcd} X \arrow[r, tail, "\eta_X"] \arrow[d, "f"'] & \mathrm{Ex}(X) \arrow[d, "\mathrm{Ex}(f)"] \\ Y \arrow[r, tail, "\eta_Y"] & \mathrm{Ex}(Y) \end{tikzcd}$$

and the 2-of-3 property it follows that $\mathrm{Ex}(f)$ is a topological weak equivalence if and only if f is one. Finally, Sd takes the generating acyclic cofibrations into acyclic cofibrations, thus Ex preserves Kan fibrations. (See Kan [12] or Goerss–Jardine [6] for careful details.)

Proposition 2.1. *Let $SSet \underset{\mathrm{Ex}^n}{\overset{\mathrm{Sd}^n}{\leftrightarrows}} SSet$ be the n-fold iteration of the simplicial subdivision – extension adjunction, and let $c_k : \partial\Delta_k \hookrightarrow \Delta_k$ be the set of generating cofibrations for $SSet$. (By convention, set Ex^0, Sd^0 to be the identity.) In $SSet$, define*

- cof_n *to be the closure under pushouts, transfinite compositions and retracts of the set of morphisms* $\mathrm{Sd}^n(c_k)$
- W *to be the class of topological weak equivalences*
- fib_n *to be the class of morphisms f such that $\mathrm{Ex}^n(f)$ is a Kan fibration.*

Then $\langle \mathsf{cof}_n, \mathsf{W}, \mathsf{fib}_n \rangle$ form a Quillen model structure on $SSet$.

Proof. Define (for the moment) W^{-1} to be the class of maps f such that $\mathrm{Ex}^n(f) \in \mathsf{W}$. Since Ex^n preserves Kan fibrations, topological weak equivalences and arbitrary filtered colimits, it follows from the small object argument that cof_n, W^{-1} and fib_n define a Quillen model structure on $SSet$. (See Hirschhorn [8] or Hovey [9] for the statement of 'creating model structures by right adjoints' in the context of cofibrantly generated model categories.) But $\mathsf{W}^{-1} = \mathsf{W}$ since $\mathrm{Ex}(f)$ is a topological weak equivalence if and only if f is one. □

Remark. As far as homotopy model theory is concerned, the crux is not so much sd being a subdivision as being a *singular functor*, that is to say, functor $\Delta \xrightarrow{s} SSet$ whose values are (weakly) contractible simplicial sets, inducing an adjunction with the above properties. There's no shortage of such singular functors, yet I am not aware of an easy argument that any of them induces a countable properly decreasing chain of 'axiomatic cofibrations'. At any rate, sd does:

Theorem 2.2. *For the model structures defined in Prop. 2.1, one has strictly monotone chains of inclusions*

$$\mathsf{fib}_0 \subsetneqq \mathsf{fib}_1 \subsetneqq \mathsf{fib}_3 \subsetneqq \ldots \subsetneqq \mathsf{fib}_n \subsetneqq \ldots$$

resp.

$$\mathsf{cof}_0 \supsetneqq \mathsf{cof}_1 \supsetneqq \mathsf{cof}_3 \supsetneqq \ldots \supsetneqq \mathsf{cof}_n \supsetneqq \ldots$$

Proof. Since Ex preserves Kan fibrations, the inclusion $\mathsf{fib}_n \subseteq \mathsf{fib}_{n+1}$ is automatic, and that implies $\mathsf{cof}_n \supseteq \mathsf{cof}_{n+1}$. The strictness follows from

Proposition 2.3. *For any $n \in \mathbb{N}$, there exists a simplicial set X such that $\mathrm{Ex}^n(X)$ does not satisfy the Kan extension condition, but $\mathrm{Ex}^{n+1}(X)$ does.*

The proof is preceded by two lemmas. The first one states, roughly, that in the n^{th} barycentric subdivision of a triangle, pairs of points on the boundary cannot be connected by interior paths shorter than 2^n. (This will be responsible for non-injectivity of a certain graph with respect to certain graph maps.) The second lemma states an analogue of this for the n^{th} simplicial subdivision of the simplex Δ_n. We then exhibit the required counterexample X: it is $R_\infty(\operatorname{Sd}^n \Lambda_2^0)$, where R_∞ is the canonical fibrantization functor for the model structure fib_n, and the horn Λ_2^0 is Δ_2 minus its (non-degenerate) 2-simplex and 0^{th} face. (See Conj. 3.1 for another guess at where counterexamples may come from.)

Lemma 2.4. *Let x and y be vertices of the n^{th} barycentric subdivision of a triangle with vertices A, B, C. Suppose x lies on the side AB and y on the side AC of the triangle, $x \neq A$ and $y \neq A$. Let p be an edge path connecting A and B. Suppose p does not pass through the vertex A. Then p contains at least 2^n edges.*

This is an example of the statement for $n = 2$:

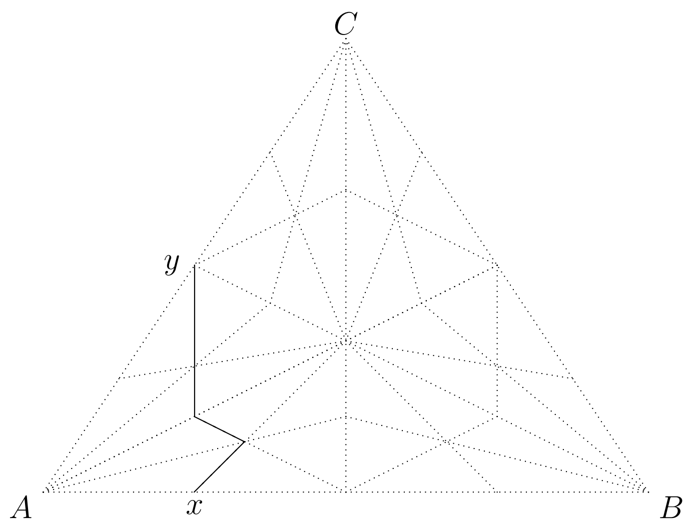


FIG. 1. To get from side AB to side AC, avoiding the vertex A,
you need an edge path of length at least 2^2 in the twice-subdivided triangle ABC.

The proof of this lemma (which is an inductive partitioning argument) is postponed. Now, for vertices x, y of a simplicial set, write $d(x, y)$ for their edge distance, that is to say, the least length of a *possibly "zig-zag" edge path* connecting them. (All simplicial sets considered below will be connected.) If x, y are vertices of X, and $X \xrightarrow{f} Y$ is a map of simplicial sets, note that

$$d(x, y) \geqslant d(f(x), f(y)).$$

As far as edge-paths are concerned, the difference between simplicial subdivision and barycentric subdivision of the standard simplices and their subcomplexes is that the edges of a simplicial set are oriented, and each vertex of a simplicial set carries a degenerate edge beginning and ending there. Neither of these affects edge distances, and in the next lemma, if so desired, one is entitled to think in terms of simplicial complexes.

Lemma 2.5. *Let x and y be vertices of $\mathrm{Sd}^n(\partial\Delta_k)$, thought of as simplicial subset of $\mathrm{Sd}^n(\Delta_k)$. If $d(x,y) < 2^n$ in $\mathrm{Sd}^n(\Delta_k)$, then the distance of x and y in $\mathrm{Sd}^n(\partial\Delta_k)$ equals their distance in $\mathrm{Sd}^n(\Delta_k)$.*

Proof. (a) Suppose there is a top-dimensional face $i : \Delta_{k-1} \hookrightarrow \Delta_k$ of our k-simplex such that $\mathrm{Sd}^n(\Delta_{k-1})$ contains both x and y. There is a retraction $r : \Delta_k \to \Delta_{k-1}$ in $SSet$ (a degeneracy 'dual' to i), whence a retraction $\mathrm{Sd}^n(r)$; by the above remark, the distance of x and y in $\mathrm{Sd}^n(\partial\Delta_k)$ then cannot be greater than their distance in $\mathrm{Sd}^n(\Delta_k)$.

(b) If no face of Δ_k contains both x and y, then, without loss of generality, assume that x lies on the face opposite the vertex $[0]$, y lies on the face opposite the vertex $[1]$, and neither lies on the intersection of these faces, the (codimension 2) face $\mathcal{F}$ with vertices $[2], [3], \ldots, [k]$. Consider a distance-minimizing path p in $\mathrm{Sd}^n(\Delta_k)$ between x and y. If p contains a vertex F on the subdivided face $\mathcal{F}$, then the argument of part (a) can be applied separately to the paths XF and FY to deduce that a distance-minimizing edge path between x and y can proceed on $\mathrm{Sd}^n(\partial\Delta_k)$, as claimed.

(c) The missing case is when the distance-minimizing path p avoids $\mathcal{F}$. We show that any such path must be of length 2^n at least, contradicting our assumption that $d(x,y) < 2^n$.

Consider the simplicial collapsing map $\Delta_k \xrightarrow{c} \Delta_2$ corresponding to the monotone map that sends $[0]$ to $[0]$, $[1]$ to $[1]$, and $[i]$ to $[2]$ for $i \geqslant 2$. Under the map $\mathrm{Sd}^n(c)$, x and y are sent into vertices of $\mathrm{Sd}^n(\Delta_2)$, x lying on the side opposite the vertex $[0]$, y lying on the side opposite the vertex $[1]$, and p will become an edge path connecting them that avoids the vertex $[2]$. By Lemma 2.4, $\mathrm{Sd}^n(p)$ has at least length 2^n, whence so does p. □

For any simplicial set U, define $R_\infty(U)$ to be the colimit of the chain

$$R_0(U) \to R_1(U) \to R_2(U) \to R_3(U) \to \ldots$$

where $R_0(U) = U$ and $R_{j+1}(U)$ arises from $R_j(U)$ by pushing on all $n+1$-times subdivided horn filling conditions

$$\begin{tikzcd} \mathrm{Sd}^{n+1}(\Lambda^i_k) \arrow[r] \arrow[d, tail] & R_j(U) \\ \mathrm{Sd}^{n+1}(\Delta_k) & \end{tikzcd}$$

that exist at that stage. By Quillen's small object argument, $R_\infty(U)$ has the right lifting property with respect to the set of maps $\mathrm{Sd}^{n+1}(\Lambda^i_k) \to \mathrm{Sd}^{n+1}(\Delta_k)$. Adjointly, $\mathrm{Ex}^{n+1}(R_\infty(U))$ is a Kan complex. Set $U = \mathrm{Sd}^n(\Lambda^0_2)$. We will exhibit a specific lifting problem with respect to an n-times subdivided horn inclusion that $X = R_\infty(\mathrm{Sd}^n(\Lambda^0_2))$ *fails*; that is to say, $\mathrm{Ex}^n(X)$ is not a Kan complex.

The lifting problem will be

$$\begin{tikzcd} \mathrm{Sd}^n(\Lambda^0_2) \arrow[r, "\text{canonical}"] \arrow[d, tail, "i"] & R_\infty(\mathrm{Sd}^n(\Lambda^0_2)) \\ \mathrm{Sd}^n(\Delta_2) \arrow[ur, dotted, "(?)"'] & \end{tikzcd}$$

$\mathrm{Sd}^n(\Lambda^0_2)$ is precisely a zig-zag of length 2^{n+1}. Call its extreme vertices x and y. (x and y are thus the vertices of Δ_2 that bound the edge missing in Λ^0_2.) Note that $d(i(x), i(y)) = 2^n$ in $\mathrm{Sd}^n(\Delta_2)$. If a lift $(?)$ existed, then it would have to exist into $R_j(\mathrm{Sd}^n(\Lambda^0_2))$ for some finite j already, since $\mathrm{Sd}^n(\Delta_2)$ is (simplicially) finite. So, letting r_j denote the canonical map $\mathrm{Sd}^n(\Lambda^0_2) \to R_j(\mathrm{Sd}^n(\Lambda^0_2))$, to prove the impossibility of a lift, it suffices to prove

$$d(r_j(x), r_j(y)) = 2^{n+1} \text{ in } R_j(\mathrm{Sd}^n(\Lambda^0_2)) \text{ for all } j \geqslant 0.$$

This is true for $j = 0$; now use induction. $R_{j+1}(\mathrm{Sd}^n(\Lambda^0_2))$ arises from $R_j(\mathrm{Sd}^n(\Lambda^0_2))$ via simultaneous pushouts of the type

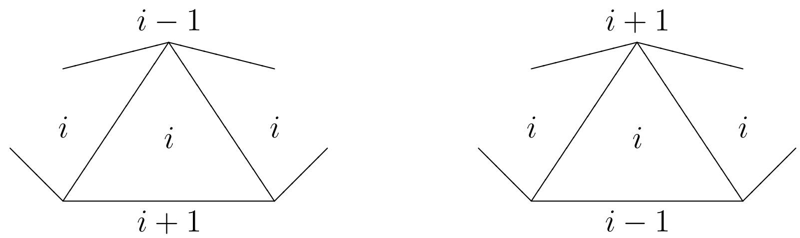
$$\begin{tikzcd} \mathrm{Sd}^{n+1}(\Lambda^i_k) \arrow[r] \arrow[d, tail] & R_j(\mathrm{Sd}^n(\Lambda^0_2)) \arrow[d] \\ \mathrm{Sd}^{n+1}(\Delta_k) \arrow[r] & R_{j+1}(\mathrm{Sd}^n(\Lambda^0_2)) \end{tikzcd}$$

Let a, b be any vertices of $R_j(\mathrm{Sd}^n(\Lambda^0_2))$. In $R_{j+1}(\mathrm{Sd}^n(\Lambda^0_2))$, possibly new paths have been pushed on that connect a and b, but by Lemma 2.5, if $d < 2^{n+1}$, paths of length d are attached only between a, b whose distance in $R_j(\mathrm{Sd}^n(\Lambda^0_2))$ is d or less. Therefore, if $d(a, b) \leqslant 2^{n+1}$, the distance of a and b in $R_j(\mathrm{Sd}^n(\Lambda^0_2))$ equals their distance in $R_{j+1}(\mathrm{Sd}^n(\Lambda^0_2))$. In particular, by the induction hypothesis, $d(r_j(x), r_j(y)) = 2^{n+1} = d(r_{j+1}(x), r_{j+1}(y))$.

To finish the proof of Prop. 2.3, we still need to prove Lemma 2.4. Let us return to the language of planar figures. By induction on n, we will show that the 6^n triangles in the n^{th} barycentric subdivision of ABC can be assigned into 2^n disjoint classes (which we will call 'rays' and label with the integers from 1 through 2^n) such that

(1) Side AB (other than the vertex A itself) lies on ray 1; side AC (other than the vertex A) lies on ray 2^n.
(2) Let T be one of the 6^n small triangles. Suppose T belongs to ray i and does not contain the vertex A. Then either $(2a)$ one edge of T lies on the common boundary of ray i and ray $i+1$ (for some $1 \leqslant i \leqslant 2^n$) and its opposite vertex lies on the common boundary of ray i and ray $i-1$ or $(2b)$ one edge of T lies on the common boundary of ray i and ray $i-1$ (for some $1 \leqslant i \leqslant 2^n$) and its opposite vertex lies on the common boundary of ray i and ray $i+1$. The interior of the other two edges of T, in both cases, will belong to ray i.

(To avoid having to state separate cases for $i = 0$ and $i = 2^n$, let us agree that the side AB belongs to ray 0, and side AC belongs to ray $2^n + 1$.)



type $(2a)$ type $(2b)$

From (1) and (2) it follows that an interior edge of the subdivided triangle, if it does not contain the vertex A, either lies on the common boundary of ray i and ray $i+1$ (for some $1 \leqslant i \leqslant 2^n$), or spans ray i (so that one of its endpoints belongs to ray $i-1$ and ray i, and the other endpoint belongs to ray i and ray $i+1$). To get from point x on the side AB to point y on the side AC, avoiding vertex A, a path must cross all 2^n rays, so must contain at least 2^n edges, as claimed.

Here are the rays for $n=1$ and $n=2$.

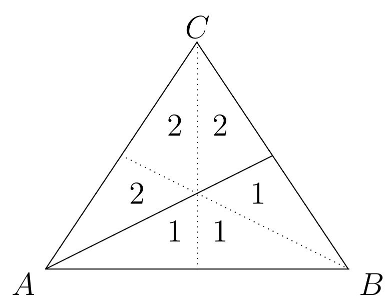


FIG. 2. The partitioning of $\mathrm{Sd}^1 \Delta_2$.

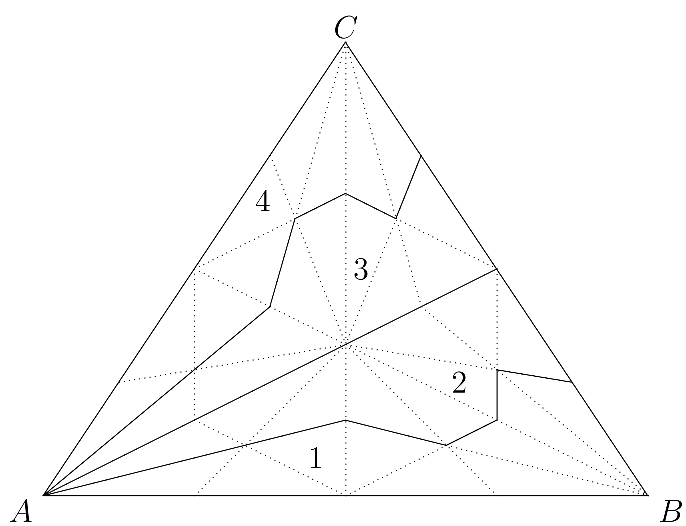
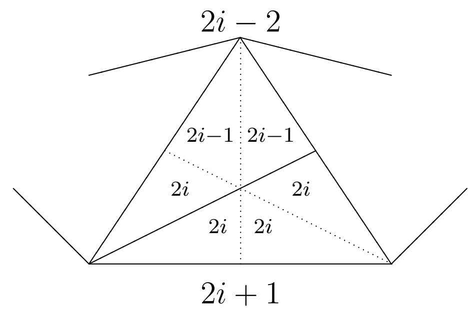
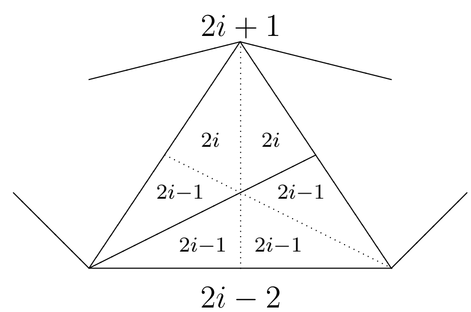


FIG. 3. The partitioning of $\mathrm{Sd}^2 \Delta_2$.
Only one triangle in each contiguous region is marked with its number i.

In general, the partitions are defined by induction. Let T be a triangle of $\mathrm{Sd}^n\,\Delta_2$, not containing the vertex A, and of the type that was denoted $(2a)$ above. Its subdivisions will then be assigned numbers



If T is of type $(2b)$, its subdivisions will be labeled



and the induction hypotheses are satisfied. (It is worthwhile to iterate the construction and observe the 'fractal boundaries' of the rays, and the self-similarity of the local patterns arising.)

As for the triangles in $\mathrm{Sd}^n\,\Delta_2$ that contain the vertex A, forming a fan around A, they are numbered consecutively from 1 (at side AB) to 2^n (at side AC); this is compatible with the subdivisions of type $(2a)$ and $(2b)$.

This, then, finishes the proof of Lemma 2.4, and also of Prop. 2.3, so of the main theorem. $\square$

One can show (see Prop. 3.3 below) that the standard simplices belong to fib_n for $n > 0$. (Δ_k itself is a Kan complex only for $k = 0$.) On the other hand, for $n > 0$ it will no longer be true that every simplicial set is cofibrant.

3. Fibrancy and nerves of categories

In his groundbreaking [17], Thomason proved that the categorification-nerve adjunction $Cat \overset{\mathsf{cat}}{\underset{N}{\leftrightarrows}} SSet$ creates a model structure on Cat, Quillen equivalent to spaces, from the one on $SSet$ we denoted $\langle \mathsf{cof}_2, \mathsf{W}, \mathsf{fib}_2 \rangle$. It follows that it creates one from $\langle \mathsf{cof}_n, \mathsf{W}, \mathsf{fib}_n \rangle$ for any $n \geqslant 2$; but it does not formally follow that the fibrancy classes of these model structures on Cat are distinct. It is tempting to believe that they *are*. That is implied by the $n \geqslant 2$ cases of the following

Conjecture 3.1. For any n, there exist categories $\mathcal{C}$ (even monoids) such that $\mathrm{Ex}^n(N\mathcal{C})$ is not a Kan complex, but $\mathrm{Ex}^{n+1}(N\mathcal{C})$ is.

This happens to be true for $n = 0$, as we recall below, but the $n \geqslant 1$ cases of Conj. 3.1 have quite a different feel.

Proposition 3.2. *For a small category $\mathcal{C}$, $N\mathcal{C}$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.*

This is classical; a proof can be found in e.g. Lee [14].

Proposition 3.3. *For a small category $\mathcal{C}$, $\mathrm{Ex}(N\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ possesses a left calculus of fractions with respect to itself.*

This is Latch–Thomason–Wilson [13], remark 5.8. Note that for a category, being a groupoid amounts to injectivity with respect to two functors in the category of (small) categories: these ensure the possibility of left and right "division". Similarly, the property of a category "possessing a left calculus of fractions with respect to itself" amounts to injectivity with respect to both of the following inclusions between finite diagrams:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

(Composition rules for arrows are omitted, but see Gabriel–Zisman [5].)

One can think of $\mathcal{C}$ "possessing a left calculus of fractions with respect to itself" as an approximation to its being a groupoid; the morphisms of $\mathcal{C}[\mathcal{C}^{-1}]$, while not actual arrows, are representable by equivalence classes of zig-zags of length 2. If one takes $\mathcal{C}$ to have a single object (i.e. to be a monoid), then for it to have a left calculus of fractions means that it satisfies the left Ore conditions; in a certain way, it is close to being a group.

Now if $\mathcal{C}$ is a groupoid, then $N\mathcal{C}$ is homotopy equivalent to the disjoint union of Eilenberg–MacLane spaces $K(\pi, 1)$ corresponding to its vertex groups. By a theorem of Dwyer and Kan [4], if $\mathcal{C}$ possesses a left or right (more generally, homotopy left or right) calculus of fractions with respect to all its morphisms, then the localization map $\mathcal{C} \to \mathcal{C}[\mathcal{C}^{-1}]$ induces a weak equivalence on nerves. By putting all these facts together, one sees that for $n \leqslant 1$ the following is true: if $\mathcal{C}$ is a small category such that $\mathrm{Ex}^n(N\mathcal{C})$ is a Kan complex, then $N\mathcal{C}$ is weakly equivalent to a disjoint union of Eilenberg–MacLane spaces (i.e. has vanishing homotopy groups above dimension 1).

For any given $n \geqslant 2$ however, by Thomason's result, the range of $\mathrm{Ex}^n(N\mathcal{C})$ will include Kan complexes within *all* homotopy types. It follows that for $n \geqslant 2$, the fibrancy of $\mathrm{Ex}^n(N\mathcal{C})$ cannot be characterized with the help of finitely many lifting conditions between small categories — unlike the cases of $n = 0$ and $n = 1$, groupoids and left Ore categories. To be sure, $\mathrm{Ex}^n(N\mathcal{C})$ is a Kan complex if and only if the category $\mathcal{C}$ is injective with respect to a certain countably infinite family of maps between finite posets, namely, $\mathsf{cat}\big(\mathrm{Sd}^n(\Lambda^i_k \hookrightarrow \Delta_k)\big)$. This suggests that Conj. 3.1 is still very "simplicial"; perhaps one can prove it by looking carefully at composability of edges in $\mathrm{Sd}^n\,\Delta_k$.

To sum up, whenever one transports via right adjoints "exotic" fibration classes such as fib_n from $SSet$ to small categories, higher groupoids, simplicial universal algebras, etc., or sheafifies them [2], it needs to be checked whether new fibrations are created. An example when this does *not* happen is across the adjunction

$$Top \underset{\mathrm{Sing}}{\overset{|-|}{\leftrightarrows}} SSet$$

owing to the fact that the geometric realization of a subdivided simplex is homeomorphic to the original. Maybe (compactly generated) topological spaces and weak equivalences possess an extremal fibration class.

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