Cellular objects and Shelah's singular compactness theorem

Logic Colloquium 2015 Helsinki

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Aug 7, 2015

how it began

Theorem (Schreier, 1926): Every subgroup of a free group is free.

Also true for abelian groups, abelian p-groups, and several equational varieties of k-algebras, for a field k:
Lie algebras (Witt), commutative (non-associative) algebras (Shirshov), magmas (Kurosh), ...

Remark: there is still no complete characterization of these varieties known.

Conversely: Does this characterize free groups?

Question Suppose G is an infinite group such that all subgroups of G of cardinality less than G, are free. (We will call such a group G *almost free*.) Is G free?

Answer No. Eklof (1970) gave examples of non-free but almost free abelian groups (and Mekler, of non-abelian groups) of cardinality $\aleph_n$, for any $n \in \mathbb{N}^+$.

On the other hand, Higman (1951) proved that for singular κ of cofinality ω_0, an almost free group of cardinality κ is necessarily free. Hill (1970): for singular κ of cofinality ω_0, an almost free abelian group of cardinality κ is necessarily free. Hill (1974): this also holds for singular κ of cofinality ω_1.

Shelah: Singular Cardinal Compactness

Shelah (1974) proved three related statements, each of which has the form:

Let κ be singular and S a structure of size κ. If all substructures of S of size less than κ have property $\mathcal{P}$, then S itself has property $\mathcal{P}$.

(1) structure $=$ abelian group; $\mathcal{P} =$ free
(2) structure $=$ graph; $\mathcal{P} =$ having coloring number $\leqslant \mu$
(3) structure $=$ set of countable sets; $\mathcal{P} =$ having a transversal.

In example (2), a graph G is defined to have *coloring number* $\leqslant \mu$ if the vertices of G can be well-ordered so that every vertex is connected to $< \mu$ vertices preceding it in the ordering.

Subsequently, many other examples added, chiefly from commutative algebra: completely decomposable modules, Q-filtered modules ...

what is Singular Cardinal Compactness *really*?

Not an instance of compactness in any classical sense (e.g. for propositional or first-order logic, or compact cardinals)

- Shelah (1974) axiomatizes when his proof works
- Hodges (1982), building on Shelah (unpublished), gives another, elegant axiomatization
- Eklof (2006): SCC is about *an abstract notion of "free"*

What is "free" about a graph with coloring number less than μ?

cells: topological origin

Let

- $D_n = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leqslant 1\}$ be the closed unit n-ball
- $\partial D_n = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| = 1\}$ be the unit n-1-sphere
- $\partial D_n \xrightarrow{i_n} D_n$ the inclusion.

A pushout p in the category of topological spaces

$$\begin{tikzcd} \partial D_n \arrow[r] \arrow[d, "i_n"] & X \arrow[d, "p"] \\ D_n \arrow[r] & Y \end{tikzcd}$$

is called *attaching an n-cell*.

cellular maps: topological origin

Let α be a well-ordered set and $F : \alpha \to Top$ a functor. Suppose

- for every $\beta < \alpha$, the map $F(\beta) \to F(\beta + 1)$ is attaching an n-cell (for some $n \in \mathbb{N}$, depending on β)
- for limit $\beta < \alpha$, the functor F is smooth at β.

Then $F(0) \to \operatorname{colim} F$ is called a *relatively cellular map*.
If $F(0)$ is empty, $\operatorname{colim} F$ is called a *cellular space*.

Cellular spaces are topological generalizations of the geometrically more restricted notions of 'simplicial complex' and 'CW-complex'.

I-cellular maps

Let $\mathcal{C}$ be a cocomplete category and I a class of morphisms. The class of *I-cellular maps* is defined as the closure under isomorphisms (in the category of morphisms of $\mathcal{C}$) of well-ordered smooth colimits of pushouts of elements of I.

An object X is I-cellular if the map $\varnothing \to X$ from the initial object to X is I-cellular.

This notion was isolated around 1970 by Quillen, Kan, Bousfield etc. It has proved very handy in homotopical algebra, in constructing weak factorization systems and homotopy model categories.

Note that (with rare exceptions) a cellular map has many cell decompositions, without any preferred / canonical / functorial one.

I-cellular maps: example 1

Let $\mathcal{C}$ be an equational variety of algebras and their homomorphisms (e.g. groups, abelian groups, R-modules $\ldots$). Let $A_\varnothing$ be the free algebra on the empty set, $A_\bullet$ the free algebra on a singleton and let $I = \{A_\varnothing \to A_\bullet\}$.

Then I-cellular objects are the same as free algebras.

I-cellular maps: example 2

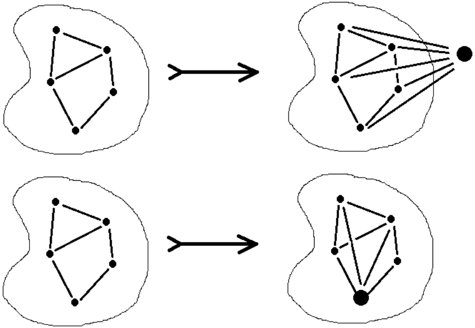
Fix a ring R and let S be a set of R-modules. Let $I = \{0 \to M \mid M \in S\}$.

Then I-cellular objects are the same as I-decomposable modules.

(This example would make sense in any category with coproducts.)

I-cellular maps: cartoon of example 3

consider the two families of inclusions of graphs



I-cellular maps: example 3

Work in the category of graphs and graph homomorphisms.
Let μ be a cardinal and let I consist of all inclusions of graphs

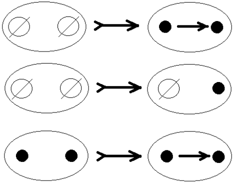
$$\langle V, E\rangle \to \langle V \cup \bullet,\ E \cup (V \times \bullet) \cup (\bullet \times V)\rangle$$

where the set V of vertices has cardinality $< \mu$ and $\bullet$ is a singleton (we take a representative set of these morphisms).

Then I-cellular objects are precisely graphs having coloring number $\leqslant \mu$. An I-cell decomposition of a graph is the same data as a well-ordering of vertices satisfying Shelah's criterion: each vertex is connected to $< \mu$ vertices preceding it.

I-cellular maps: example 4

Work in the category of directed bipartite graphs, i.e. triples $\langle U, V, E\rangle$ where U, V are sets and E is a relation from A to B. Let I consist of the three morphisms



where $\varnothing$ is the empty set, $\bullet$ a singleton. Pushouts by these "create a disjoint edge", "create a vertex in the second partition", "create an edge between existing vertices" respectively. I-cellular objects are precisely directed bipartite graphs $\langle U, V, E\rangle$ that posses at least one transversal $U \rightarrowtail V$: a cell decomposition of a graph provides data for a transversal, and vice versa.

cellular SCC: cleanest version

Let $\mathcal{B}$ be a locally finitely presentable category and I a set of morphisms. Let $X \in \mathcal{B}$ be an object whose size $\|X\|$ is singular. If all subobjects of X of size less than $\|X\|$ are I-cellular, then X itself is I-cellular.

cellular SCC: clarifying remarks

- The *size* of an object X of a locally presentable category is the cardinal predecessor of the least cardinal κ such that X is κ-presentable. (The least such κ will indeed be a successor cardinal.) This notion of size is intrinsic to the category, and coincides with the naive notion of 'cardinality of the underlying set' in all cases of interest.
- It suffices to have 'enough' subobjects of X to be cellular for the conclusion to hold.
- Could have a proper class of generating maps as long as the size of their domains is bounded from above.
- The size of X should be big enough (above the Löwenheim–Skolem number).

Singular Cardinal Compactness: Cellular version

Theorem (B.–Rosický, 2014)

Let $\mathcal{B}$ be a locally finitely presentable category, μ a regular uncountable cardinal and I a class of morphisms with μ-presentable domains. Let $X \in \mathcal{B}$ be an object with $\max\{\mu, \mathrm{card}(\mathrm{fp}\,\mathcal{B})\} < \|X\|$. Assume

(i) $\|X\|$ is a singular cardinal

(ii) there exists $\phi < \|X\|$ such that for all successor cardinals κ^+ with $\phi < \kappa^+ < \|X\|$, there exists a dense κ^+-filter of $\mathrm{Sub}(X)$ consisting of I-cellular objects.

Then X is I-cellular.

the one that got away

Theorem (Hodges, 1982)

Let k be a field and K/k a field extension. Suppose K is κ-generated over k for some singular cardinal κ, and for all intermediate extensions L between k and K, if L is λ-generated over k for some regular cardinal λ, then L is a purely transcendental extension of k. Then K itself is a purely transcendental extension of k.

This is a corollary of Hodges's axiomatization of Shelah's proof.

It is *not* a case of the cellular version of singular compactness.

Singular Cardinal Compactness: Functorial version

Theorem (B.–Rosický, 2014)

Let $\mathcal{A}$ be an accessible category with filtered colimits, $\mathcal{B}$ a finitely accessible category and $F : \mathcal{A} \to \mathcal{B}$ a functor preserving filtered colimits. Let $X \in \mathcal{B}$ be an object with $\max\{\mu_F, \mathrm{card}(\mathrm{fp}\,\mathcal{B})\} < \|X\|$. Assume

(i) $\|X\|$ is a singular cardinal

(ii) there exists $\phi < \|X\|$ such that for all successor cardinals κ^+ with $\phi < \kappa^+ < \|X\|$, the image of F contains a dense κ^+-filter of $\mathrm{Sub}(X)$

(iii) F-structures extend along morphisms.

Then X is in the image of F.

functorial SCC: clarifying remarks

- Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. We say that *F-structures extend along morphisms* if, given any morphism $g : X \to Y$ and object U of $\mathcal{A}$, together with an isomorphism $i : F(U) \to F(X)$ in $\mathcal{B}$, there exists a morphism $f : U \to V$ and isomorphism $j : F(V) \to F(Y)$ such that

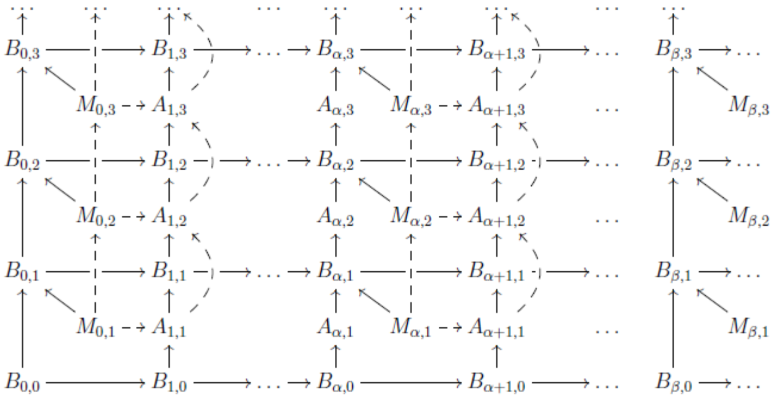
$$\begin{tikzcd} F(U) \arrow[r, "F(f)"] \arrow[d, "i"] & F(V) \arrow[d, "j"] \\ F(X) \arrow[r, "F(g)"] & F(Y) \end{tikzcd}$$

  commutes.
- An object of $\mathcal{B}$ is 'in the image' of the functor F if it is isomorphic to $F(X)$ for some X in $\mathcal{A}$.

about the proof

- Assume first the target category is $\mathrm{Pre}(\mathcal{C})$, the category of presheaves on a small category $\mathcal{C}$. Adapt the set-based proof of Hodges (1982). Fairly easy since the poset of subobjects of any object of $\mathrm{Pre}(\mathcal{C})$ is a complete distributive lattice.
- Characterize the categories $\mathcal{B}$ that posses a nice enough embedding $\mathcal{B} \to \mathrm{Pre}(\mathcal{C})$ into a presheaf category so that one can deduce the conclusion for $F : \mathcal{A} \to \mathcal{B}$, given that it holds for the composite $F : \mathcal{A} \to \mathcal{B} \to \mathrm{Pre}(\mathcal{C})$.
- The functorial version implies both the cellular version and Hodges's version.

the big picture



questions

Both the cellular and functorial versions of SCC are machines for churning out conditionals of the following type:

" If X is almost free and its size is singular, then it is free. "

Such a statement could be rather 'worthless' for two reasons:

- There exists no almost free X whose size is singular.
- If X is almost free then it is free (regardless of size).

bother!

Theorem (Hodges, 1982)

Let k be a field and K/k a field extension. Suppose K is κ-generated over k for some singular cardinal κ, and for all intermediate extensions L between k and K, if L is λ-generated over k for some regular cardinal λ, then L is a purely transcendental extension of k. Then K itself is a purely transcendental extension of k.

- Is this a statement about the empty set?

That is, are there *any* such K/k?

the combinatorics of (counter)examples

Concerning groups, abelian groups, μ-colorable graphs, set transversals:

We've uniformized the proofs in *singular* characteristics (where almost free objects are free).

• Is there a way to unify the construction of 'paradoxical' i.e. almost free but non-free objects in *regular* characteristics?

Note that these constructions are fairly few and scattered. They seem to work best for $\aleph_n$ for finite n, or under $V = L$ or similar set-theoretic assumptions.