ISOPERIMETRIC INEQUALITIES AND THE FRIEDLANDER–MILNOR CONJECTURE

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Abstract. We prove that Friedlander's generalized isomorphism conjecture on the cohomology of algebraic groups, and hence the Isomorphism Conjecture for the cohomology of the complex algebraic Lie group $\mathsf{G}(\mathbb{C})$ made discrete, are equivalent to the existence of an isoperimetric inequality in the homological bar complex of $\mathsf{G}(F)$, where F is the algebraic closure of a finite field.

Introduction

For any topological group G, let G^δ denote G with the same group structure, but considered as a discrete space. The continuous homomorphism $G^\delta \xrightarrow{\text{id}} G$ induces a map of classifying spaces $BG^\delta \xrightarrow{i} BG$. In [11], Milnor stated:

Milnor's conjecture on the homology of Lie groups made discrete: If G is a Lie group, then i induces isomorphisms[1]

$$H^n_{top}(BG, \mathbb{Z}/l) \to H^n_{top}(BG^\delta, \mathbb{Z}/l) = H^n(G^\delta, \mathbb{Z}/l)$$

for any prime l. The motivation for this (at first read, no doubt, surprising) conjecture was

Friedlander's conjecture [6]**:** Let k be an algebraically closed field and l a prime distinct from the characteristic of k. Let G_k be an algebraic group over k. Then the natural map of group schemes $G(k)_k \to G_k$ induces an isomorphism

$$H^n_{\acute{e}t}(BG_k, \mathbb{Z}/l) \to H^n_{\acute{e}t}(BG_k(k), \mathbb{Z}/l) = H^n(G(k), \mathbb{Z}/l).$$

Here $G(k)$ is the discrete group of k-rational points of the algebraic group G_k, and the right-hand side is ordinary group cohomology. On the left, one has the étale cohomology of the simplicial scheme BG_k. The construction of the comparison map between the two is akin to turning a topological space into the discrete space made up of the set of its points, save now one turns a scheme G_k into $G(k)_k$, the coproduct of copies of the terminal k-scheme spec(k), indexed by the scheme-theoretic points of G_k.

The last chapter of Knudson [8] provides careful details and an excellent overview of the many partial results on these conjectures. In addition, the very end of section 1 of the

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[1] To ensure readability, unadorned H stands for the (co)homology of discrete groups throughout this paper, and '*top*' indicates singular (co)homology.

present paper contains historical information on their origin that the reader is encouraged to become aware of.

Friedlander–Mislin [6] proved Friedlander's conjecture over $k = \overline{\mathbb{F}}_p$, the algebraic closure of a finite field. The idea of this paper is to exploit the interaction of the homologies of the discrete group $\mathsf{G}(\overline{\mathbb{F}}_p)$ and of $\mathsf{G}(K)$ for large, algebraically closed fields K to attack more cases of Friedlander's conjecture. Here G is an integral form of a connected reductive algebraic group; such a G is assumed to have been fixed throughout. Note that its group of complex points, $\mathsf{G}(\mathbb{C})$, falls under the domain of both Milnor's and Friedlander's conjectures, and it is known that for such groups the two are equivalent.

Our main result is formulated in terms of metric properties of the bar complex for computing group homology. Let G be a discrete group and R some ring of coefficients, on which G is acting trivially. Recall that the bar complex is a functorial chain complex whose homology is $H_*(G, R)$. The module of n-chains, $C_n(G)$, is the free R-module on the basis set G^n; let d_n denote the standard boundary map $C_n(G) \xrightarrow{d_n} C_{n-1}(G)$, and $B_n(G)$ resp. $Z_n(G)$ the submodules of n-boundaries and n-cycles. Let the size $\|c\|$ of a chain $c \in C_n$ mean the number of non-zero coefficients in the expression of c as formal linear combination of elements of G^n. (If the coefficients R were a normed abelian group, one would take the sum of the absolute values of the coefficients, but throughout this paper we are concerned with prime coefficients $R = \mathbb{Z}/l$.)

The *filler norm* of a boundary $b \in B_n$ is defined as

$$\|b\|_{\mathsf{fill}} := \min \left\{ \|c\| \;\middle|\; c \in C_{n+1} \text{ such that } d_{n+1}(c) = b \right\}$$

Definition 0.1. G satisfies a *homological isoperimetric inequality for boundaries in homological degree n with coefficients R* if for all $K \in \mathbb{N}$,

$$\mathsf{isop}(K) := \sup \left\{ \|b\|_{\mathsf{fill}} \;\middle|\; b \in B_n \text{ such that } \|b\| = K \right\} < \infty$$

In words, the size of the shortest filler for a boundary b can be estimated from above in terms of the size of b itself. We call isop the *homological isoperimetric function* for G.

Fix now a G as above, prime p, homological degree n and coefficients $\mathbb{Z}/l$, $l \neq p$.

Theorem A. The following are equivalent:
- $\mathsf{G}(\overline{\mathbb{F}}_p)$ satisfies a homological isoperimetric inequality in degree n with coefficients $\mathbb{Z}/l$
- Friedlander's conjecture holds for $H^n(\mathsf{G}(k), \mathbb{Z}/l)$ for all algebraically closed fields k of characteristic p.

The characteristic zero version involves, rather than an isoperimetric function of $\mathsf{G}(\mathbb{C})^\delta$, an asymptotic isoperimetric function for $\mathsf{G}(\overline{\mathbb{F}}_p)$ as p ranges over the primes.

Theorem B. Consider the statement:
(asymp) There exists a function $\mathsf{asymp} : \mathbb{N} \to \mathbb{N}$ with the property: for each $K \in \mathbb{N}$, for all sufficiently large primes p (depending on K) one has that for all $b \in B_n(\mathsf{G}(\overline{\mathbb{F}}_p))$ with $\|b\| = K$, $\|b\|_{\mathsf{fill}} \leqslant \mathsf{asymp}(K)$.

• (asymp) implies that $H^n(\mathsf{G}(\mathbb{C})^\delta, \mathbb{Z}/l)$ satisfies Milnor's conjecture; equivalently, that Friedlander's conjecture holds for $H^n(\mathsf{G}(k), \mathbb{Z}/l)$ for all algebraically closed fields k of characteristic zero.
• The converse holds provided the homology of a maximal torus surjects on the homology of G; more precisely, if G has a maximal torus T (defined over the integers) such that for all but finitely many primes p, the inclusion $\mathsf{T}(\overline{\mathbb{F}}_p) \hookrightarrow \mathsf{G}(\overline{\mathbb{F}}_p)$ induces a surjection

$$H_n(\mathsf{T}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \twoheadrightarrow H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l).$$

This condition surfaces rather often in the study of Friedlander's conjecture, and is well-understood by a case-by-case analysis; perhaps it is enough to point out that for any G, it holds for all primes l that are large for G, i.e. that do not divide the order of the Weyl group of G (and the list of exceptional l is typically much smaller). See Section 3 for more information on (removing) this obstacle from the converse implication.

Note that (asymp) neither implies that any particular $\mathsf{G}(\overline{\mathbb{F}}_p)$ satisfies an isoperimetric inequality, nor is implied by the existence of isoperimetric functions for individual $\mathsf{G}(\overline{\mathbb{F}}_p)$ (unless those functions also happen to be uniformly bounded in p). (asymp) does imply that the uncountable group $\mathsf{G}(\mathbb{C})^\delta$ satisfies an isoperimetric inequality, but I do not know the converse.

Let k be an infinite field. Set-theoretically, the cardinality of $H_n(\mathsf{G}(k), \mathbb{Z}/l)$ is at most that of k. Friedlander's conjecture predicts that for algebraically closed k, $H_n(\mathsf{G}(k), \mathbb{Z}/l)$ is isomorphic to the finite group $H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l)$. Since there is always a surjection, this is the 'smallest' value $H_n(\mathsf{G}(k), \mathbb{Z}/l)$ can take. The last result of this paper shows that as k increases, the cardinality of $H_n(\mathsf{G}(k), \mathbb{Z}/l)$ either grows as fast as it can, or stays constant *countable*.

Theorem C. Fix G, n, l and the characteristic $p \neq l$ through which our algebraically closed fields k range (p can be a prime or zero). One of the following two possibilities obtains:
• all the $H_n(\mathsf{G}(k), \mathbb{Z}/l)$ are countable, all the groups $\mathsf{G}(k)$ possess the same isoperimetric function, and moreover every extension $k \to K$ between algebraically closed fields induces an isomorphism $H_n(\mathsf{G}(k), \mathbb{Z}/l) \xrightarrow{=} H_n(\mathsf{G}(K), \mathbb{Z}/l)$,
• *or* $H_n(\mathsf{G}(k), \mathbb{Z}/l)$ has the cardinality of k for all uncountable k.

For $p > 0$, thanks to Theorem A, the first alternative means the truth of Friedlander's conjecture. In particular, for an uncountable k of positive characteristic, $H_n(\mathsf{G}(k), \mathbb{Z}/l)$ has either the value predicted by Friedlander's conjecture, or the cardinality of k. In characteristic zero, Theorem C is far less useful; it does not restrict the values that $H_n(\mathsf{G}(\mathbb{C})^\delta, \mathbb{Z}/l)$ might take, and leaves open the possibility that $\mathsf{G}(\mathbb{C})^\delta$ possesses an isoperimetric function, though no asymptotic isoperimetric function exists for the $\mathsf{G}(\overline{\mathbb{F}}_p)$.

Terminological caveat. Isoperimetric inequalities for boundaries in the bar complex, at least with $\mathbb{Z}$ or $\mathbb{R}$ coefficients, go back in the literature to the 80's, prompted by Gromov's groundbreaking work on bounded cohomology. (See for example Matsumoto–Morita [10], who refer to the condition '$\mathsf{isop}(K) \leqslant C \cdot K$ for some constant C' as the *uniform boundary* *property.*) Isoperimetric functions for n-balls in locally finite models of $K(G,1)$ appear under the name *higher Dehn functions*; see especially Alonso–Wang–Pride [1]. These generalize the classical combinatorial Dehn function, or isoperimetric function, of finitely presented groups. Homological isoperimetric functions have also been widely considered, especially in the context of hyperbolic groups, for cycles (with $\mathbb{Z}$ or $\mathbb{R}$ coefficients) on the universal cover of suitable locally finite models of $K(G,1)$; see for example Lang [9]. It is not clear how the notions that pertain to *locally finite* models of $K(G,1)$ interact with isoperimetric inequalities *in the bar complex* — not to mention that our groups, such as $\mathsf{G}(\overline{\mathbb{F}}_p)$, are not finitely generated. In this paper, *isoperimetric inequality* is always understood in the sense of Def. 0.1.

1. Heuristic

The author discovered the relevance of isoperimetric inequalities by analyzing a well-known bridge between the algebraic closures of finite fields and uncountable algebraically closed fields. Though the proofs can be phrased without it, it is perhaps useful to give a blueprint of this bridge, as the syntactic details of the argument may otherwise conceal the simplicity of the main idea.

Let $\mathsf{P} \subset \mathbb{N}$ be an infinite set of primes, and $\mathcal{U}$ any non-principal ultrafilter on P. It is an old observation that

$$\prod_{\mathsf{P}/\mathcal{U}} \overline{\mathbb{F}}_p \approx \mathbb{C}^\delta \tag{1.1}$$

since both the ultraproduct on the left and the complex numbers (just as an untopologized field) are algebraically closed fields of characteristic zero, of the cardinality of the continuum. The $\approx$ sign is to emphasize how non-canonical the isomorphism is; it relies on Steinitz's theorem, i.e. the possibility of a set-theoretic bijection between transcendence bases (over the rationals) of the two sides.

Let G be an algebraic group defined over the integers. (1.1) extends to give an isomorphism (non-canonically, and only as discrete groups)

$$\prod_{\mathsf{P}/\mathcal{U}} \mathsf{G}(\overline{\mathbb{F}}_p) \approx \mathsf{G}(\mathbb{C})^\delta \tag{1.2}$$

The main ingredient in the proof of Friedlander's conjecture over $\overline{\mathbb{F}}_p$ is the fact, proved earlier by Friedlander and (in special cases) by Quillen, that for $p \neq l$

$$H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \approx H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l).$$

Friedlander's conjecture asserts

$$H_n(\mathsf{G}(\mathbb{C})^\delta, \mathbb{Z}/l) \approx H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l). \tag{?}$$

Supposing that the functor $H_n(-, \mathbb{Z}/l)$ commutes with ultraproducts,

$$\prod_{\mathsf{P}/\mathcal{U}} H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \approx H_n(\prod_{\mathsf{P}/\mathcal{U}} \mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \tag{?}$$

one would obtain Friedlander's conjecture:

$$
\begin{tikzcd}
\prod_{\mathsf{P}/\mathfrak{U}} H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \arrow[r, squiggly, "\sim"] \arrow[d, squiggly, "\wr"] & H_n(\prod_{\mathsf{P}/\mathfrak{U}} \mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \arrow[d, squiggly, "\wr"] \\
H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l) & H_n(\mathsf{G}(\mathbb{C})^\delta, \mathbb{Z}/l)
\end{tikzcd}
\tag{1.3}
$$

where the left-hand vertical isomorphism uses that, since the groups $H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l)$ are finite, canonically

$$\prod_{\mathsf{P}/\mathfrak{U}} H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l) = H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l).$$

Ultraproducts, while slightly tamer, are nearly as badly behaved for homological algebra as infinite products, and it is easy to see that the functor $H_n(-, \mathbb{Z}/l)$ does not in general preserve them. Nonetheless, one has a natural comparison map

$$H_n(\prod_{\mathsf{P}/\mathfrak{U}} \mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \xrightarrow{[\iota]} \prod_{\mathsf{P}/\mathfrak{U}} H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$$

Most of the work goes into understanding the kernel and image of this homomorphism. The reasons for falling back on the bar complex are its functoriality and simple syntax, which make the interaction with ultraproducts much easier to analyze. (That is also the reason for preferring to work with homology rather than cohomology.) The homomorphism $[\iota]$ turns out to be onto provided the homology of G is supported on a maximal torus (and, I conjecture, in fact always). Via (1.3), Friedlander's conjecture is seen to be equivalent to the injectivity of $[\iota]$. The condition (asymp) results from a combinatorial re-writing of this injectivity.

The positive characteristic case, Theorem A, is similar throughout but much simpler; it uses ultrapowers of $\overline{\mathbb{F}}_p$. Theorem C follows from the methods of the previous parts combined with an elementary set-theoretic observation about constructible stratifications of algebraic varieties over uncountable fields.

In homological degree $n = 1$, the isoperimetric function of any group can be understood completely in terms of its commutator width, and one can establish the main properties of the isoperimetric functions of certain groups (e.g. divisible abelian), in any homological degree, by hand. It seems to be challenging, however, to 'reverse engineer' the deep and beautiful work of Suslin [13] [14] in K-theory and homological stability that yielded (in a range of dimensions) the generalized isomorphism conjecture for GL_N and SL_N, and to say something about the isoperimetric functions associated to these groups; not to mention, of course, establishing or refuting new cases of the generalized isomorphism conjecture. The difficulty is inherent, in part, in the fact that our proof of Theorem B is non-constructive, i.e. proceeds by contradiction. It is worth noting, however, the similarity between Suslin's "universal cycles" and the isoperimetric condition in the bar complex. These threads will be pursued elsewhere; our goal here is just to prove Theorems A, B and C.

Historical remarks. I am indebted to an anonymous referee for bringing the following to my attention.

The situation of Theorem B of this paper — groups of the form $\mathsf{G}(\mathbb{C})$ where G is an integral form of a connected reductive algebraic group — is exactly what E. Friedlander considered and discussed with many people during his stay in Princeton (1970–1975), when he first stated and investigated his conjecture. Milnor then generalized the conjecture to an arbitrary Lie group with finitely many connected components. (Cf. the third sentence of Milnor [11]: "This paper is organized around the following conjecture which was suggested to the author by E. Friedlander at least in the complex case.") In the literature, the conjecture in this general form is called the "Friedlander–Milnor Conjecture" (or, as Milnor calls it, the "Isomorphism Conjecture").

In other words, that part of Milnor's conjecture to which this paper has relevance is due to Friedlander. The extension of the Isomorphism Conjecture to algebraically closed base fields other than $\mathbb{C}$, the "Generalized Isomorphism Conjecture", is due to Friedlander alone.

E. Friedlander informs me that he and Charles Miller attempted to use ultraproducts to attack his conjecture in the 70's.

The first published use of ultraproducts is this context is due to Jardine [7]. Other than the underlying idea of building uncountable algebraically closed fields as ultraproducts of algebraic closures of finite fields, his methods and conclusions are distinct from ours.

2. Ultraproducts of the bar complex

The goal of this section is to construct a comparison homomorphism from the homology of an ultraproduct of groups to the ultraproduct of their homologies, and to give a necessary and sufficient condition for it to be injective resp. surjective. We only use elementary results on ultraproducts and model theory in this paper, all contained in the textbook Bell–Slomson [2].

The map is constructed via one particular device for computing group homology, the bar complex. For syntactic reasons, we spell out some standard definitions in detail. Fix a ring R of coefficients, on which all groups are understood to be acting trivially. For a discrete group G, the bar complex can be thought of as the simplicial homology of the nerve of G or, alternatively, as the result of tensoring with $-\otimes_G R$ the bar resolution of R as trivial G-module. The n-chains $C_n(G)$ are the free R-module on the basis set G^n; we will write basis elements as $\langle g_1, g_2, \ldots, g_n\rangle$ or $\vec{g}$. The boundary mapping $C_n \xrightarrow{d_n} C_{n-1}$ is defined on basis elements by

$$\begin{aligned}\langle g_1, g_2, \ldots, g_n\rangle \mapsto \langle g_2, \ldots, g_n\rangle - \langle g_1 g_2, \ldots, g_n\rangle + \langle g_1, g_2 g_3, \ldots, g_n\rangle - \ldots \\ +(-1)^{n-1}\langle g_1, g_2, \ldots, g_{n-1}g_n\rangle + (-1)^n\langle g_1, g_2, \ldots, g_{n-1}\rangle\end{aligned}$$

(Here $d_1(\langle g_1\rangle) = \langle\rangle - \langle\rangle = 0\cdot\langle\rangle$, where the empty tuple $\langle\rangle$ is the generator of $C_0(G)$, and $C_{-1}(G)$ is by definition zero.) If z is a cycle, we write $[z]$ for the homology class it represents.

Let G_λ, $\lambda \in \Lambda$, be a set of discrete groups. Let $\mathcal{U}$ be an ultrafilter on Λ, and let $\mathcal{G}$ denote the corresponding ultraproduct $\prod_{\Lambda/\mathcal{U}} G_\lambda$. If $\phi(\lambda)$ is a mathematical statement containing the parameter λ ranging over Λ, we will abbreviate as

$$\mathcal{U} \models \phi$$

the statement "the set of $\lambda \in \Lambda$ for which $\phi(\lambda)$ is true, belongs to $\mathcal{U}$". (Remark: only the variable λ will be used in this role. Though the notation is suggestive, it is meant to be just a typographical device. In particular, ϕ will be typically phrased in the meta-language, and is not necessarily assumed to be equivalent to a first-order formula in the language of rings and groups.)

Consider now the ultraproduct (as R-modules) $\prod_{\Lambda/\mathcal{U}} C_n(G_\lambda)$. One has an R-linear map

$$C_n(\mathcal{G}) \xrightarrow{\iota} \prod_{\Lambda/\mathcal{U}} C_n(G_\lambda)$$

defined on basis elements as follows: given $\vec{g} \in \mathcal{G}^n$, choose representatives $\{\vec{g}_\lambda \in G_\lambda^n \mid \lambda \in \Lambda\}$ for it; the collection

$$\big\{1 \cdot \vec{g}_\lambda \mid \lambda \in \Lambda\big\}$$

gives a well-defined element $\iota(\vec{g})$ of the ultraproduct $\prod_{\Lambda/\mathcal{U}} C_n(G_\lambda)$.

Proposition 2.1. *ι is injective.*

Proof. Let X be an arbitrary set, and fix a k-tuple $\langle r_1, r_2, \dots, r_k\rangle$ of elements of R. To say that the formal expression $r_1 \cdot x_1 + r_2 \cdot x_2 + \dots + r_k \cdot x_k$ (where the x_i are thought of as variable, ranging over X) equals 0 in the free R-module with basis X amounts to a first-order formula

$$\bigvee_{\substack{I_1 \sqcup I_2 \sqcup \dots \sqcup I_p \\ = \{1,2,\dots,k\}}} \bigwedge_{q=1}^{p} \bigwedge_{i,j \in I_q} x_i = x_j$$

where the disjunction is over all partitions of $\{1, 2, \dots, k\}$ into subsets $\{I_1, I_2, \dots, I_p\}$ in such a way that $\sum_{i \in I_q} r_i = 0$ for each $q = 1, 2, \dots, p$. Call this formula $\theta(x_1, x_2, \dots, x_k)$. Given $c = \sum r_i \, \vec{g}_i$ in $C_n(\mathcal{G})$ and representatives $\{\vec{g}_{i,\lambda} \in G_\lambda^n \mid \lambda \in \Lambda\}$ for $\vec{g}_i$, the following are equivalent:

$$\begin{aligned}
&\iota(c) = 0 \text{ in } \prod_{\Lambda/\mathcal{U}} C_n(G_\lambda) \\
&\Leftrightarrow \mathcal{U} \models \quad \sum r_i \, \vec{g}_{i,\lambda} = 0 \text{ in } C_n(G_\lambda) \\
&\Leftrightarrow \mathcal{U} \models \quad \theta(\vec{g}_{1,\lambda}, \vec{g}_{2,\lambda}, \dots, \vec{g}_{k,\lambda}) \\
&\Leftrightarrow \theta(\vec{g}_1, \vec{g}_2, \dots, \vec{g}_k) \text{ holds in } C_n(\mathcal{G}) \\
&\Leftrightarrow c = \sum r_i \, \vec{g}_i = 0 \text{ in } C_n(\mathcal{G})
\end{aligned}$$

by Łós's theorem. ∎

$\prod_{\Lambda/\mathcal{U}} C_n(G_\lambda)$ can be equipped with a 'boundary' map $\hat{d}_n : \prod_{\Lambda/\mathcal{U}} C_n(G_\lambda) \to \prod_{\Lambda/\mathcal{U}} C_{n-1}(G_\lambda)$, which is simply the ultraproduct of the boundary mappings connecting the individual $C_n(G_\lambda)$. It is therefore linear and one checks (via basis elements) that $\iota(d_n(c)) = \hat{d}_n(\iota(c))$ for $c \in \prod_{\Lambda/\mathcal{U}} C_n(G_\lambda)$.

Now for each homology class in $H_n(\mathcal{G}, R)$ take a representing cycle $z = \sum r_i\, \vec{g}_i$ from $Z_n(\mathcal{G})$. Choosing representatives $\{\vec{g}_{i,\lambda} \in G^n_\lambda \,|\, \lambda \in \Lambda\}$ for each $\vec{g}_i$, one has that

$$\mathcal{U} \models \quad \sum r_i\, \vec{g}_{i,\lambda} \in Z_n(G_\lambda)$$

hence $\iota(z)$ represents an element in (the ultraproduct as R-modules) $\prod_{\Lambda/\mathcal{U}} H_n(G_\lambda, R)$. This element is independent of the choice of representative z taken in its homology class. Indeed, if z' is another such, then $z - z' = d_{n+1}(c)$ for some $c \in C_{n+1}(\mathcal{G})$, implying

$$\mathcal{U} \models \quad \iota(z) \text{ and } \iota(z') \text{ are homologous in } Z_n(G_\lambda).$$

One therefore obtains a map

$$H_n(\mathcal{G}, R) \xrightarrow{[\iota]} \prod_{\Lambda/\mathcal{U}} H_n(G_\lambda, R) \tag{2.1}$$

Remark 2.2. It is also true that $\hat{d}_n \circ \hat{d}_{n+1} = 0$ and $\ker \hat{d}_n / \operatorname{im} \hat{d}_{n+1}$ is canonically isomorphic to $\prod_{\Lambda/\mathcal{U}} H_n(G_\lambda, R)$, but we won't need this.

Using the fact that the algebraic structure on homology classes is definable directly on cycle representatives, one sees that $[\iota]$ is R-linear. We wish to understand the kernel and image of $[\iota]$. This turns out to be more tedious for the case of an infinite R, nor does that case have relevance to Friedlander's conjecture. (See the Appendix of Milnor [11] for an investigation of $H^{top}_*(BG^\delta)$ with rational or real coefficients.) Henceforth we *assume the cardinality of R to be finite*, and introduce a partial inverse to ι.

Recall that the *size* $\|c\|$ of an element of a free R-module with specified basis is the number of basis elements occurring with non-zero coefficients.

Definition 2.3. An element $\hat{c}$ of $\prod_{\Lambda/\mathcal{U}} C_n(G_\lambda)$ is said to be $\mathcal{U}$-*uniformly bounded* (or simply *bounded*) if there exists $K < \infty$ such that for some (equivalently, all) representatives $\{c_\lambda \in C_n(G_\lambda) \,|\, \lambda \in \Lambda\}$ of $\hat{c}$,

$$\mathcal{U} \models \ \|c_\lambda\| \leqslant K$$

It is immediate that for any $c \in C_n(\mathcal{G})$, $\iota(c)$ is bounded; bounded chains form a submodule of $\prod_{\Lambda/\mathcal{U}} C_n(G_\lambda)$; and $\hat{d}(\hat{c})$ is bounded if $\hat{c}$ is so.

Let $\hat{c} \in \prod_{\Lambda/\mathcal{U}} C_n(G_\lambda)$ be bounded in size by K. Choose representatives $\{c_\lambda \in C_n(G_\lambda) \,|\, \lambda \in \Lambda\}$ of $\hat{c}$, and write each c_λ with $|c_\lambda| \leqslant K$ in *some* way as an *ordered* sum of basis elements, $c_\lambda = \sum_{i=1}^{l_\lambda} r_i\, \vec{g}_{i,\lambda}$, $l_\lambda \leqslant K$. This allows one to define a map recording 'coordinates'

$$\{c_\lambda \in C_n(G_\lambda) \text{ such that } |c_\lambda| \leqslant K\} \xrightarrow{\text{coor}} R^{\leqslant K}$$

(where $R^{\leqslant K}$ is the set of ordered tuples from R of size at most K) by sending $c_\lambda = \sum_{i=1}^{l_\lambda} r_i\,\vec{g}_{i,\lambda}$ to $\mathrm{coor}(c_\lambda) \stackrel{\mathrm{def}}{=} \langle r_1, r_2, \ldots, r_{l_\lambda}\rangle$. The map f from the appropriate element of $\mathcal{U}$ to $R^{\leqslant K}$ defined by $\lambda \mapsto \mathrm{coor}(c_\lambda)$ partitions a member of the ultrafilter into finitely many disjoint subsets via $f^{-1}(t)$ as t ranges over the elements of $R^{\leqslant K}$. So for exactly one tuple t_0 will the set $f^{-1}(t_0)$ belong to $\mathcal{U}$. Let that t_0 be $\langle r_1, r_2, \ldots, r_l\rangle$ and write U for $f^{-1}(t_0)$; then one has that for all $\lambda \in U$,

$$c_\lambda = \sum_{i=1}^{l} r_i\,\vec{g}_{i,\lambda}$$

for well-defined $r_i \in R$, $\vec{g}_{i,\lambda} \in G_\lambda^n$. For a fixed i, the collection $\{\vec{g}_{i,\lambda} \mid \lambda \in U\}$ (extended by arbitrary $\vec{g}_{i,\lambda}$ for $\lambda \in \Lambda \setminus U$, if necessary) can be thought of as an element $\vec{g}_i$ in the ultraproduct $\mathcal{G}^n$. Introduce the notation

$$\tau(\hat{c}) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{l} a_i\,\vec{g}_i \in C_n(\mathcal{G}).$$

Obviously $\iota(\tau(\hat{c})) = \hat{c}$. But since ι is injective, ι and τ must be inverse bijections between $C_n(\mathcal{G})$ and the submodule of $\prod_{\Lambda/\mathcal{U}} C_n(G_\lambda)$ consisting of bounded chains. In particular, τ is independent of the choices made, R-linear, and $d_n(\tau(\hat{c})) = \tau(\hat{d}_n(\hat{c}))$ for any bounded $\hat{c} \in \prod_{\Lambda/\mathcal{U}} C_n(G_\lambda)$.

Corollary 2.4. *Call a class in $\prod_{\Lambda/\mathcal{U}} H_n(G_\lambda, R)$* bounded *if it can be represented by a bounded cycle in $\prod_{\Lambda/\mathcal{U}} C_n(G_\lambda)$. Bounded classes form a submodule of $\prod_{\Lambda/\mathcal{U}} H_n(G_\lambda, R)$, which equals the image of $[\iota]$.*

Corollary 2.5. *$[\iota]$ is surjective if and only if there exists $K < \infty$ such that for $\mathcal{U}$-most λ, every homology class in $H_n(G_\lambda, R)$ contains a cycle of size at most K.*

Proposition 2.6. *$[\iota]$ is injective if and only if the following holds:*

($\bigstar$) *For any bounded $\hat{b} \in \prod_{\Lambda/\mathcal{U}} C_n(G_\lambda)$, if $\mathcal{U} \models b_\lambda \in B_n(G_\lambda)$, then there exists a bounded $\hat{c} \in \prod_{\Lambda/\mathcal{U}} C_{n+1}(G_\lambda)$ such that $\mathcal{U} \models\ b_\lambda = d_{n+1}(c_\lambda)$.*

Proof. Suppose ($\bigstar$) holds. Let $b \in Z_n(\mathcal{G})$ be a cycle representative of a homology class in $H_n(\mathcal{G}, R)$ that is sent to zero by $[\iota]$. That is to say, $\mathcal{U} \models \iota(b) \in B_n(G_\lambda)$. $\iota(b)$ is bounded, hence by assumption there exists a bounded $\hat{c} \in \prod_{\Lambda/\mathcal{U}} C_{n+1}(G_\lambda)$ such that $\hat{d}_{n+1}(\hat{c}) = \iota(b)$. But then $d_{n+1}(\tau(\hat{c})) = \tau(\hat{d}_{n+1}(\hat{c})) = \tau(\iota(b)) = b$. Thence b is a boundary, so represents the zero homology class.

Conversely, assume $[\iota]$ is injective, and let $\hat{b}$ be bounded and $\mathcal{U}$-almost everywhere a boundary. Then $d_n(\tau(\hat{b})) = \tau(\hat{d}_n(\hat{b})) = \tau(0) = 0$, so $\tau(\hat{b})$ represents a homology class in $H_n(\mathcal{G}, R)$. $[\iota]([\tau(\hat{b})]) = 0$ since $\iota(\tau(\hat{b})) = \hat{b}$ is $\mathcal{U}$-almost everywhere a boundary by assumption. By the injectivity of $[\iota]$, there must exist $c \in C_{n+1}(\mathcal{G})$ such that $d_{n+1}(c) = \tau(\hat{b})$. $\iota(c)$ is bounded and

$$\hat{d}_{n+1}(\iota(c)) = \iota(d_{n+1}(c)) = \iota(\tau(\hat{b})) = \hat{b}$$

so (★) is satisfied. □

In effect, this says that (★) holds if and only if $[\hat{z}] \mapsto [\tau(\hat{z})]$ is the inverse bijection to $[z] \mapsto [\iota(z)]$ between $H_n(\mathcal{G}, R)$ and the module of bounded homology classes.

3. From Friedlander's conjecture to isoperimetric functions

The key result that makes the previous section applicable to Friedlander's conjecture is due to Friedlander [5] [6], based on work of Quillen [12]:

Theorem 3.1. *Let $p \neq l$ be primes. $H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$ is isomorphic to $H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l)$.*

This latter group is finite (and known, as a function of G, n and l). Let us introduce the notation $|H_{\mathsf{G},n,l}|$ for the common value of the cardinalities of the groups $H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$, $l \neq p$, and $H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l)$.

Lemma 3.2. $\operatorname{card} H_n(\mathsf{G}(\mathbb{C})^\delta, \mathbb{Z}/l) \geqslant |H_{\mathsf{G},n,l}|$. *Friedlander's conjecture holds for the Lie group $\mathsf{G}(\mathbb{C})$ if and only if* $\operatorname{card} H_n(\mathsf{G}(\mathbb{C})^\delta, \mathbb{Z}/l) = |H_{\mathsf{G},n,l}|$.

Proof. Both statements follow from the theorem of Milnor [11] that $H_n(G^\delta, \mathbb{Z}/l) \xrightarrow{i} H_n^{top}(BG, \mathbb{Z}/l)$ is surjective for any Lie group G with finitely many components. □

The next observation (also well-known) contains Lemma 3.2; they are repeated just to emphasize the parallel. Note that Milnor's proof is purely topological and applies to real Lie groups as well.

Lemma 3.3. *Let k be an algebraically closed field of characteristic $p \neq l$. Then* $\operatorname{card} H_n(\mathsf{G}(k), \mathbb{Z}/l) \geqslant |H_{\mathsf{G},n,l}|$. *Friedlander's conjecture holds for G_k if and only if* $\operatorname{card} H_n(\mathsf{G}(k), \mathbb{Z}/l) = |H_{\mathsf{G},n,l}|$.

Proof. By a theorem of Friedlander and Mislin, the map $H^n_{\acute{e}t}(B\mathsf{G}_k, \mathbb{Z}/l) \to H^n(\mathsf{G}(k), \mathbb{Z}/l)$ concerned in the generalized isomorphism conjecture is injective. Now $H^n_{\acute{e}t}(B\mathsf{G}_k, \mathbb{Z}/l) \approx H^n_{\acute{e}t}(B\mathsf{G}_{\overline{\mathbb{F}}_p}, \mathbb{Z}/l) \approx H^n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \approx H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$ by virtue of the invariance of étale cohomology under algebraically closed field extensions and the truth of the generalized isomorphism conjecture over $\overline{\mathbb{F}}_p$; and if $H^n(\mathsf{G}(k), \mathbb{Z}/l)$ or $H_n(\mathsf{G}(k), \mathbb{Z}/l)$ is finite, then they are isomorphic. These facts imply both parts. □

We are now ready to prove one direction of Theorem A:

Proposition 3.4. *If Friedlander's generalized isomorphism conjecture holds for $H_n(\mathsf{G}(k), \mathbb{Z}/l)$ for all algebraically closed fields k of characteristic p, then $\mathsf{G}(\overline{\mathbb{F}}_p)$ satisfies an isoperimetric inequality in homological degree n with coefficients $\mathbb{Z}/l$.*

Proof. Consider any non-principal ultrafilter $\mathcal{U}$ on any countable set Λ, and write P for the ultrapower $\prod_{\Lambda/\mathcal{U}} \overline{\mathbb{F}}_p$. P is an algebraically closed field of characteristic p, of the cardinality of the continuum. As G is first-order definable in the language of rings, $\prod_{\Lambda/\mathcal{U}} \mathsf{G}(\overline{\mathbb{F}}_p)$ is canonically isomorphic to $\mathsf{G}(P)$. Apply the comparison homomorphism (2.1):

$$H_n(\mathsf{G}(P), \mathbb{Z}/l) \xrightarrow{[\iota]} \prod_{\Lambda/\mathcal{U}} H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \tag{3.1}$$

Since $H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$ is finite, the right-hand side is isomorphic to $H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$. $[\iota]$ is surjective, with a splitting induced by the inclusion $\overline{\mathbb{F}}_p \hookrightarrow P$ (which can also be thought of as the diagonal $\overline{\mathbb{F}}_p \to \prod_{\Lambda/\mathcal{U}} \overline{\mathbb{F}}_p$); see Cor. 2.5.

Proceed by contradiction. Fix some size $K \in \mathbb{N}$, and let B_K denote the set of boundaries $b \in B_n(\mathsf{G}(\overline{\mathbb{F}}_p))$ with $\|b\| = K$. Suppose

$$\left\{ \|b\|_{\mathsf{fill}} \;\middle|\; b \in B_K \right\} \subseteq \mathbb{N}$$

were unbounded. Then there would exist an infinite subset $\Lambda \subseteq B_K$ such that for any infinite $U \subseteq \Lambda$,

$$\left\{ \|b\|_{\mathsf{fill}} \;\middle|\; b \in U \right\}$$

is still unbounded. This Λ will serve as the index set for a comparison map of the type (3.1); the non-principal ultrafilter $\mathcal{U}$ on Λ can be arbitrary.

One has a tautologous element $\hat{s} \in \prod_{\Lambda/\mathcal{U}} B_n(G_\lambda)$, associating to $\lambda \in \Lambda$ itself, since $\Lambda \subseteq B_n(\mathsf{G}(\overline{\mathbb{F}}_p))$. By construction, $\hat{s}$ is $\mathcal{U}$-bounded in size by K. The existence of a $\mathcal{U}$-bounded $\hat{c} \in \prod_{\Lambda/\mathcal{U}} C_{n+1}(G_\lambda)$ such that $\mathcal{U} \models\; s_\lambda = d_{n+1}(c_\lambda)$ would mean that there exist a constant $K_1 < \infty$ and $U \in \mathcal{U}$ such that for all $b \in U$, $\|b\|_{\mathsf{fill}} \leqslant K_1$. Since $\mathcal{U}$ is non-principal, U would be an infinite subset of Λ, contradicting the choice of Λ. Therefore property ($\bigstar$) of Prop. 2.6 fails, and $[\iota]$ cannot be injective.

But that means that the cardinality of $H_n(\mathsf{G}(P), \mathbb{Z}/l)$ exceeds that of $H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$, and so (cf. Lemma 3.3) the generalized isomorphism conjecture fails over P, the (unique) algebraically closed field of characteristic p that has the cardinality of the continuum. $\square$

The other direction of Theorem A follows easily from the comparison map (3.1), injectivity condition ($\bigstar$), and a theorem of Friedlander-Mislin stating that if the generalized isomorphism conjecture holds for one algebraically closed field of infinite transcendence degree over its prime field, then it holds for all algebraically closed fields within that characteristic. However, we prefer to give a completely elementary and self-contained proof of that direction in Section 4.

Let us turn to the hard part of Theorem B. Let P be any infinite set of primes, and $\mathcal{U}$ any non-principal ultrafilter on P. The ultraproduct of discrete groups $\prod_{\mathsf{P}/\mathcal{U}} \mathsf{G}(\overline{\mathbb{F}}_p)$ is (non-canonically) isomorphic to $\mathsf{G}(\mathbb{C}^\delta) = \mathsf{G}(\mathbb{C})^\delta$, a complex algebraic Lie group made discrete. Apply the comparison homomorphism (2.1) to the family $\mathsf{G}(\overline{\mathbb{F}}_p)$, $p \in \mathsf{P}$:

$$H_n(\prod_{\mathsf{P}/\mathcal{U}} \mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \xrightarrow{[\iota]} \prod_{\mathsf{P}/\mathcal{U}} H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \tag{3.2}$$

For all $p \neq l$, a fortiori for all but finitely many $p \in \mathsf{P}$, $H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$ is isomorphic to the finite group $H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l)$, so the right-hand side of (3.2) is isomorphic to $H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l)$. Though the argument is analogous to characteristic p, a point has to be overcome in order to deduce the condition (asymp) from Friedlander's conjecture.

Lemma 3.5. *Suppose* G *has a maximal torus* T *(defined over the integers) such that for all but finitely many primes* p*, the inclusion* $\mathsf{T}(\overline{\mathbb{F}}_p) \hookrightarrow \mathsf{G}(\overline{\mathbb{F}}_p)$ *induces a surjection*

$$H_n(\mathsf{T}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \twoheadrightarrow H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l).$$

Then the $[\iota]$ *of* (3.2) *is surjective, for any non-principal ultrafilter* $\mathfrak{U}$ *on any infinite set of primes* P.

Proof. We wish to prove the following: there exists a bound $f_{\mathsf{G}}(n, l) < \infty$ such that for all but finitely many primes p, each homology class $\alpha \in H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$ has a cycle representative $z_{p,\alpha} \in Z_n(\mathsf{G}(\overline{\mathbb{F}}_p))$ with $\|z_{p,\alpha}\| \leqslant f_{\mathsf{G}}(n, l)$. By Cor. 2.5, this implies (and is in fact equivalent to) the conclusion.

Such a bound exists for the torus $\mathsf{T} = GL_1 \times \cdots \times GL_1$ of rank r. Assume $p \neq l$. Identifying the l^{th}-power roots of unity in $\overline{\mathbb{F}}_p$ with $\mathbb{Z}/l^\infty$, one gets an injection $\mathbb{Z}/l^\infty \to \overline{\mathbb{F}}_p^\times$ such that $(\mathbb{Z}/l^\infty)^r \to \mathsf{T}(\overline{\mathbb{F}}_p)$ induces isomorphism on $H_*(-, \mathbb{Z}/l)$. One can take $f_{\mathsf{T}}(n, l) = \max\{\|z_1\|, \|z_2\|, \ldots, \|z_N\|\}$ where the cycles z_i span the (finite) group $H_n((\mathbb{Z}/l^\infty)^r, \mathbb{Z}/l)$.

Under the assumption that the homology of a maximal torus surjects on the homology of G, one can take $f_{\mathsf{G}}(n, l) = f_{\mathsf{T}}(n, l)$ for the torus T of the same rank as G. $\square$

Discussion. For the sake of completeness, let us recall how 'cheap' this assumption is. There are several well-known ways to investigate $H_n(\mathsf{T}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \twoheadrightarrow H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$.

The functoriality of Friedlander's isomorphism $H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \approx H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l)$ is subtle, as it depends on an embedding of the Witt vectors of $\overline{\mathbb{F}}_p$ in $\mathbb{C}$. However, by making choices simultaneously for G and its split maximal torus T, one obtains a commutative diagram

$$\begin{tikzcd}
H_n(\mathsf{T}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \arrow[r] \arrow[d, "\approx"] & H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \arrow[d, "\approx"] \\
H_n^{top}(B\mathsf{T}(\mathbb{C}), \mathbb{Z}/l) \arrow[r] & H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l)
\end{tikzcd}$$

On the topological side, one has a surjection $H_n^{top}(B\mathsf{T}(\mathbb{C}), \mathbb{Z}/l) \twoheadrightarrow H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l)$ when (for example) l is prime to the order of the Weyl group of G; one way to see this is to approximate the classifying space of a Lie group by manifolds, and use Becker–Gottlieb transfer. See Feshbach [4].

For Chevalley groups G, one can also argue purely group-theoretically. Suppose $l \neq p$, $l \nmid |\mathsf{W}|$, and let the p-power q be such that $\mathbb{F}_q$ contains l^{th} roots of unity. By a theorem of Chevalley [3], there exists a split maximal torus T of G such that $\mathsf{T}(\mathbb{F}_q)$ contains a Sylow

l-subgroup of $\mathsf{G}(\mathbb{F}_q)$. (This is because $\big[\mathsf{G}(\mathbb{F}_q) : \mathsf{T}(\mathbb{F}_q)\big]$ will be prime to l.) Therefore $\mathsf{T}(\mathbb{F}_q) \hookrightarrow \mathsf{G}(\mathbb{F}_q)$ induces a surjection

$$H_n(\mathsf{T}(\mathbb{F}_q), \mathbb{Z}/l) = H_n(\mathrm{Syl}_l \mathsf{T}(\mathbb{F}_q), \mathbb{Z}/l) = H_n(\mathrm{Syl}_l \mathsf{G}(\mathbb{F}_q), \mathbb{Z}/l) \twoheadrightarrow H_n(\mathsf{G}(\mathbb{F}_q), \mathbb{Z}/l).$$

Let $\mathbb{F}_q$ be cofinal in $\overline{\mathbb{F}}_p$ such that $q \equiv 1 \pmod{l}$. Since the tori T can be chosen compatibly, there results a surjection $H_n(\mathsf{T}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \twoheadrightarrow H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$.

If l is a torsion prime for G, one need not have a surjection $H_n(\mathsf{T}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \twoheadrightarrow H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$. Nonetheless, a Sylow subgroup of $\mathsf{G}(\mathbb{F}_q)$ is always contained in the normalizer of a torus, which is an extension of a torus by the Weyl group. By making compatible choices as $\mathbb{F}_q$ increases to $\overline{\mathbb{F}}_p$, one obtains a surjection $H_n(N_{\mathsf{T}}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \twoheadrightarrow H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$ with a short exact sequence $\mathbf{1} \to \mathsf{T}(\overline{\mathbb{F}}_p) \to N_{\mathsf{T}}(\overline{\mathbb{F}}_p) \to \mathsf{W} \to \mathbf{1}$.

The Lyndon-Hochschild-Serre spectral sequence for this extension has the form

$$H_i\big(\mathsf{W}, H_j(\mathsf{T}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)\big) \Rightarrow H_{i+j}(N_{\mathsf{T}}(\overline{\mathbb{F}}_p), \mathbb{Z}/l).$$

Since all homology groups involved are finite, the spectral sequence converges (in any total degree) in a finite number of steps. In principle at least, one can check that the homology of W has cycle representatives (with twisted coefficients) whose size is bounded independently of p; analyzing the differentials in the spectral sequence, presumably so does $H_n(N_{\mathsf{T}}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$ and, eventually, $H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$ — for all G, n, p and l. At this stage, it does not seem worthwhile to spell out these details for exceptional l. (Recall that the implication from the existence of an asymptotic isoperimetric function to the truth of Friedlander's conjecture, to be proved in section 4, holds unconditionally.)

We return to the proof of Theorem B now.

Proposition 3.6. *Let G, n, l be such that the conclusion of Lemma 3.5 holds. Friedlander's conjecture for $H_n(\mathsf{G}(\mathbb{C})^\delta, \mathbb{Z}/l)$ implies that the asymptotic isoperimetric function of Theorem B exists.*

Proof. Fix some $K \in \mathbb{N}$. Write $B_K(p)$ for the set of boundaries $b \in B_n(\mathsf{G}(\overline{\mathbb{F}}_p))$ with $\|b\| = K$. By contradiction, assume: for all $K_1 \in \mathbb{N}$, there exist infinitely many primes p such that $\sup\big\{\|b\|_{\mathsf{fill}} \mid b \in B_K(p)\big\} > K_1$.

That would allow one to find an infinite set $\mathsf{P} = \{p_0, p_1, p_2, \dots\}$ of primes and for each $p \in \mathsf{P}$ some boundary $b_p \in B_K(p)$ with the property that for any infinite subset $U \subseteq \mathsf{P}$, the set $\{\|b_p\|_{\mathsf{fill}} \mid p \in U\}$ is unbounded. (Let p_0 and b_{p_0} be arbitrary, and having found p_n, pick $p_{n+1} \notin \{p_0, p_1, \dots, p_n\}$ and $b_{p_{n+1}} \in B_K(p_{n+1})$ such that $\|b_{p_{n+1}}\|_{\mathsf{fill}} > \|b_{p_n}\|_{\mathsf{fill}}$.) Let this P be the infinite set of primes with which the comparison homomorphism (3.2) is constructed. If Friedlander's conjecture holds, then by Lemma 3.2, the two sides of (3.2) have the same finite cardinality. So if $[\iota]$ is surjective, it must be injective too, and condition ($\bigstar$) of Prop. 2.6 must be satisfied. On the other hand, for the $\mathcal{U}$-bounded element $\{p \mapsto b_p\} \in \prod_{\mathsf{P}/\mathcal{U}} B_n(\mathsf{G}(\overline{\mathbb{F}}_p))$ there cannot exist a $\mathcal{U}$-bounded $\hat{c} \in \prod_{\mathsf{P}/\mathcal{U}} C_{n+1}(\mathsf{G}(\overline{\mathbb{F}}_p))$ such that $\mathcal{U} \models \ b_p = d_{n+1}(c_p)$: since $\mathcal{U}$ is non-principal, that would mean that for some infinite subset $U \subseteq \mathsf{P}$, there does exist K_1 such that for all $p \in U$, $\|b_p\|_{\mathsf{fill}} < K_1$, contradicting the choice of P.

Therefore, under Friedlander's conjecture, one can find $\mathsf{asymp}(K) = K_1 < \infty$ such that for all but finitely many primes p, $\sup\left\{\|b\|_{\mathsf{fill}} \mid b \in B_K(p)\right\} \leqslant K_1$. $\square$

4. From isoperimetric functions to Friedlander's conjecture

One can phrase the mathematics behind the other directions of Theorems A and B in two ways, different only linguistically. One is the language of constructible subsets of varieties over algebraically closed fields, Chevalley's theorem on the image of constructible sets under regular maps being constructible, base extensions between algebraically closed fields, and specialization (this is the spirit of the next section) and the other is the language of sets definable in the first-order theory of algebraically closed fields, Tarski's theorem on quantifier elimination, and the first-order Lefschetz principle. Considering the syntax of the statements involved, the second approach seems much more convenient, and that is what we will use.

Conventions. The algebraic group G defined over the integers, homological degree n, and finite ring of coefficients $R = \mathbb{Z}/l$ will be fixed once and for all. Variables will range over the algebraically closed field k; that makes the group of k-rational points $\mathsf{G}(k)$ and the group operations on $\mathsf{G}(k)$ first-order expressible in the language of rings. Observe that none of "chain", "cycle" and "boundary" in the bar complex are first-order expressible. However, *for any given choice of the bounds K, K_1*, each of

$$\text{“ chain } c \in C_n(\mathsf{G}(k)) \text{ with } \|c\| = K \text{ ”}$$
$$\text{“ cycle } z \in Z_n(\mathsf{G}(k)) \text{ with } \|z\| = K \text{ ”}$$
$$\text{“ boundary } b \in B_n(\mathsf{G}(k)) \text{ with } \|b\| = K \text{ and } \|b\|_{\mathsf{fill}} = K_1 \text{ ”}$$

is first-order expressible. (Code a chain $c \in C_n$ of size K as $K \cdot l$ many n-tuples of elements of $\mathsf{G}(k)$, exploit the first-order definition of the bar differential d_n and the fact that the equality of two expressions that are unordered formal R-linear combinations is first-order.)

For any $K, K_1, K_2 \in \mathbb{N}$, consider the sentence Φ_{K,K_1,K_2}

" For every $b \in C_n(\mathsf{G}(k))$ with $\|b\| = K$, if there exists $u \in C_{n+1}(\mathsf{G}(k))$ with $\|u\| = K_1$ such that $d_{n+1}(u) = b$, then there exists $c \in C_{n+1}(\mathsf{G}(k))$ with $|c| \leqslant K_2$ such that $d_{n+1}(c) = b$. "

By Tarski's theorem, Φ_{K,K_1,K_2} either holds in all algebraically closed fields k of a given characteristic, or none. But the countable conjunction $\bigwedge_{K_1 \in \mathbb{N}} \Phi_{K,K_1,K_2}$ means precisely

$$\text{For any boundary with } \|b\| = K, \text{ one has } \|b\|_{\mathsf{fill}} \leqslant K_2.$$

Corollary 4.1. *Fix any $K \in \mathbb{N}$. As k ranges through algebraically closed fields within any given characteristic,*

$$\sup\left\{\|b\|_{\mathsf{fill}} \mid b \in B_n(\mathsf{G}(k)) \text{ such that } \|b\| = K\right\} \in \mathbb{N} \cup \infty$$

stays the same.

So it makes sense to talk of "the isoperimetric function of G in characteristic p", p a prime or zero, provided this supremum is finite for all $K \in \mathbb{N}$.

Proposition 4.2. *If* G *satisfies an isoperimetric inequality in characteristic* p *(*p *a prime or zero) and* $H_n(\mathsf{G}(k), R)$ *is finite for one particular algebraically closed* k *of that characteristic, then the groups* $H_n(\mathsf{G}(k), R)$ *are isomorphic for all algebraically closed* k *of characteristic* p*.*

Proof. Let k_1 be such that the cardinality of $H_n(\mathsf{G}(k), R)$, as k varies over algebraically closed fields in characteristic p, is minimal at $k = k_1$. Write $|H|$ for that least cardinality; by assumption, $|H| < \infty$. For any $K \in \mathbb{N}$, set $K_1 = \max\{\mathsf{isop}(1), \mathsf{isop}(2), \ldots, \mathsf{isop}(2K)\}$ and consider the sentence

$\big(\Psi_K\big)$ " Given $z_i \in Z_n(\mathsf{G}(k))$ with $\|z_i\| \leqslant K$, $i = 1, 2, \ldots, |H| + 1$, there exist $1 \leqslant i \neq j \leqslant |H| + 1$ and $c \in C_{n+1}(\mathsf{G}(k))$ with $|c| \leqslant K_1$ such that $z_i - z_j = d_{n+1}(c)$. "

Since this is first-order and holds over $k = k_1$, it holds for all algebraically closed k of characteristic p. But the countable conjunction $\bigwedge_{K\in\mathbb{N}} \Psi_K$ just means

" Given $|H| + 1$ cycles in $Z_n(\mathsf{G}(k))$, some two of them are homologous. "

So the cardinality of $H_n(\mathsf{G}(k), R)$ is $|H|$ for all k.

Let now $k \to K$ be an extension of algebraically closed fields. The induced map $H_n(\mathsf{G}(k), R) \to H_n(\mathsf{G}(K), R)$ is injective (for example) by model completeness of algebraically closed fields: if a cycle defined over k becomes a boundary over K, then a chain responsible for its being a boundary must be definable already over k. So within characteristic p, all such maps must be isomorphisms, and $H_n(\mathsf{G}(k), R) = H_n(\mathsf{G}(k_0), R)$ where k_0 is the algebraic closure of the prime field. □

Corollary 4.3. *If* G *satisfies an isoperimetric inequality in characteristic* $p > 0$*, then Friedlander's conjecture holds in that characteristic.*

This statement (which is the other half of Theorem A) follows by the finiteness of $H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$ and Lemma 3.3. The other half of Theorem B uses the first-order Lefschetz principle:

Proposition 4.4. *If the function* (asymp) *of Theorem B exists for* G*, then Friedlander's conjecture holds.*

Proof. Fix l; for any $K \in \mathbb{N}$, set $K_1 = \max\{\mathsf{asymp}(1), \mathsf{asymp}(2), \ldots, \mathsf{asymp}(2K)\}$ and $|H|$ to be the common cardinality of the groups $H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$, $p \neq l$. Consider the sentence Ψ_K displayed above. By assumption, it holds over $k = \overline{\mathbb{F}}_p$ for infinitely many primes p. By the Lefschetz principle, it holds over all algebraically closed fields of characteristic zero. That means that, in the notation of Lemma 3.2, the cardinality of $H_n(\mathsf{G}(\mathbb{C})^\delta, \mathbb{Z}/l)$ is at most $|H_{\mathsf{G},n,l}|$. Apply Lemma 3.2. □

A similar argument shows that the function asymp must be at the same time an isoperimetric bound for the group $\mathsf{G}(\mathbb{C})^\delta$. The converse implication does not pass through the first-order Lefschetz principle, and, unlike in the case of $\overline{\mathbb{F}}_p$, no information is available regarding $H_n(\mathsf{G}(\overline{\mathbb{Q}}), \mathbb{Z}/l)$ that would make Prop. 4.2 applicable — excepting those cases when the full conjecture has already been proven!

5. Stratifying the space of cycles

The following observation has long been known in saturated model theory, but for convenience we include a proof. By *constructible subset* of a variety $\mathcal{V}$ we mean one belonging to the boolean algebra generated by Zariski-closed subsets of $\mathcal{V}$. We only consider varieties defined over some algebraically closed field k, and we identify them with their k-points.

Lemma 5.1. *Let $\mathcal{V}$ be a variety over an uncountable, algebraically closed field k. Suppose one has a collection Z_i, $i \in I$, of constructible subsets of $\mathcal{V}$ such that* $\operatorname{card} I < \operatorname{card} k$ *and*

$$Z := \bigcup_{i \in I} Z_i$$

is constructible as well. Then there exists a finite set $i_1, i_2, \dots, i_N \in I$ such that

$$Z = Z_{i_1} \cup Z_{i_2} \cup \cdots \cup Z_{i_N}.$$

One proof of Lemma 5.1 is akin to the 'cylindrical' proof of the compactness of the product of two compact topological spaces. Without loss of generality, we may assume $\mathcal{V}$ to be affine space k^n. Also without loss of generality, we may assume $Z = k^n$. (Just add the complement of Z to the original collection.)

The proof is now by induction on n. For $n = 1$, the conclusion follows since a constructible subset of k is finite or co-finite, and by the assumption $\operatorname{card} I < \operatorname{card} k$, one of the Z_i has to be co-finite. Assuming the claim holds below dimension n, write $n = r + s$ for some $0 < r, s < n$ and $k^n = A \times B$ with $A = k^r$, $B = k^s$.

For any $a \in A$, $\{a\} \times B$ is covered by its constructible subsets $Z_i \cap (\{a\} \times B)$. By the induction hypothesis for k^s, there exists a finite index set $I_a \subseteq I$ such that

$$\{a\} \times B = \bigcup_{i \in I_a} Z_i \cap (\{a\} \times B).$$

Each set defined as

$$C_a := \{x \in A \mid \text{ for all } y \in B,\ \langle x, y\rangle \in \bigcup_{i \in I_a} Z_i\}$$

forms a constructible subset of A, and their union is A. As a ranges over A, the range of I_a,

$$\{J \subseteq I \mid J = I_a \text{ for some } a \in A\}$$

(thought of as a subset of the power set of I) has cardinality less than that of k, since in fact the cardinality of all finite subsets of I equals $\operatorname{card} I < \operatorname{card} k$. By the induction hypothesis for k^r, one can find finitely many $a_1, a_2, \dots, a_N \in A$ such that $C_{a_1} \cup C_{a_2} \cup \cdots \cup C_{a_N} = A$. But that implies

$$k^n = A \times B = \bigcup_{\substack{j \in I_{a_i} \\ i = 1, 2, \dots, N}} Z_j\,.$$

Proposition 5.2. *Let k be an uncountable, algebraically closed field. If* $\operatorname{card} H_n(\mathsf{G}(k), \mathbb{Z}/l) < \operatorname{card} k$*, then $\mathsf{G}(k)$ satisfies an isoperimetric inequality.*

Proof. Having set up enough bookkeeping details, this becomes an immediate consequence of Lemma 5.1.

Bookkeeping. From here on, n-chains will be thought of as *ordered* formal linear combinations of n-tuples of group elements. Pick a representative z_α of each homology class $\alpha \in H_n(\mathsf{G}(k), \mathbb{Z}/l)$. Let $0 \in H_n(\mathsf{G}(k), \mathbb{Z}/l)$ be represented by the empty string. Fix some size K and tuple of coefficients $r_i \in \mathbb{Z}/l$, $i = 1, 2, \dots, K$. Define $\mathcal{Z}_{r_1,r_2,\dots,r_K}$ as the locus in $\mathsf{G}(k)^{nK}$ of $\big\langle\langle g_{11}, g_{12}, \dots, g_{1n}\rangle, \dots, \langle g_{K1}, g_{K2}, \dots, g_{Kn}\rangle\big\rangle$ such that $\sum_{i=1}^K r_i\langle g_{i1}, \dots, g_{in}\rangle$ is a cycle in the bar complex. Define $Z(\alpha, K_1)$ as the locus in $\mathcal{Z}_{r_1,r_2,\dots,r_K}$ of those cycles $z = \sum_{i=1}^K r_i\langle g_{i1}, \dots, g_{in}\rangle$ that satisfy $z - z_\alpha = d_{n+1}(c)$ for some chain $c \in C_{n+1}(\mathsf{G}(k))$ with $\|c\| \leqslant K_1$.

$\mathcal{Z}_{r_1,r_2,\dots,r_K}$ is an algebraic (i.e. Zariski-closed) subset of $\mathsf{G}(k)^{nK}$ and each $Z(\alpha, K_1)$ is a constructible subset of $\mathcal{Z}_{r_1,r_2,\dots,r_K}$. Indeed, to say that $z = \sum_{i=1}^K r_i\langle g_{i1}, \dots, g_{in}\rangle$ is a cycle is to say that at least one of a finite number of possible cancellation patterns occurs among the $(n+1)K$ many n-tuples that make up $d_{n+1}(z)$. Each such cancellation pattern is a system of equalities — stated purely in terms of the group multiplication on $\mathsf{G}(k)$ — among the group elements g_{ij}. Each $Z(\alpha, K_1)$ is a constructible subset of $\mathcal{Z}_{r_1,r_2,\dots,r_K}$. Indeed, fix a K_1-tuple of coefficients for the $n+1$-chain c, and denote by $[d_{n+1}(c) = z - z_\alpha]$ the locus in $\mathsf{G}(k)^{(n+1)K_1+nK}$ of those pairs $\langle c, z\rangle$, $c \in C_{n+1}(\mathsf{G}(k))$, $z \in \mathcal{Z}_{r_1,r_2,\dots,r_K}$ that satisfy $d_{n+1}(c) = z - z_\alpha$. $\big[d_{n+1}(c) = z - z_\alpha\big]$ is Zariski-closed, and $Z(\alpha, K_1)$ is a finite union (as the K_1-tuple of coefficients varies) of images of $\big[d_{n+1}(c) = z - z_\alpha\big]$ under the projection $\mathsf{G}(k)^{(n+1)K_1+nK} \xrightarrow{\mathrm{pr}} \mathsf{G}(k)^{nK}$.

Obviously $Z(\alpha, K_1) \subseteq Z(\alpha, K_2)$ for $K_1 < K_2$, and $Z(\alpha, K_1) \cap Z(\beta, K_2) = \emptyset$ for $\alpha \neq \beta$. Since

$$\bigcup_{\substack{\alpha \in H_n(\mathsf{G}(k), \mathbb{Z}/l)\\ K_1 \in \mathbb{N}}} Z(\alpha, K_1) = \mathcal{Z}_{r_1,r_2,\dots,r_K}$$

the lemma implies that one has a finite disjoint decomposition

$$\mathcal{Z}_{r_1,r_2,\dots,r_K} = \bigsqcup_{j \in J} Z(\alpha_j, K_j). \tag{5.1}$$

One (at most) of these homology classes α_j, say α_0, is the zero one. That means that every boundary of the form $\sum_{i=1}^K r_i\langle g_{i1}, \dots, g_{in}\rangle$ possesses a filler of length at most K_0. Letting the tuple $r_1, r_2, \dots, r_K$ range over its (finitely many!) possibilities, one obtains a finite value for $\mathsf{isop}(K)$. □

Together with Cor. 4.1, the next proposition completes the proof of Theorem C of the introduction.

Proposition 5.3. *If* $\operatorname{card} H_n(\mathsf{G}(k_0), \mathbb{Z}/l) < \operatorname{card} k_0$ *for one uncountable, algebraically closed field* k_0*, then, within the characteristic of* k_0*, the groups* $H_n(\mathsf{G}(k), \mathbb{Z}/l)$ *are countable for all algebraically closed* k*, and every extension* $k_1 \to k_2$ *between algebraically closed fields induces an isomorphism* $H_n(\mathsf{G}(k_1), \mathbb{Z}/l) \xrightarrow{=} H_n(\mathsf{G}(k_2), \mathbb{Z}/l)$.

Proof. For brevity, let us introduce the notation $\mathrm{dist}(z_1, z_2)$ for $\|z_1 - z_2\|_{\mathsf{fill}}$ whenever z_1, z_2 are homologous cycles. Note that $\mathrm{dist}(z_1, z_3) \leqslant \mathrm{dist}(z_1, z_2) + \mathrm{dist}(z_2, z_3)$. Over the field k_0, one has a finite disjoint decomposition (5.1) of the space of cycles $\mathcal{Z}_{r_1,r_2,\ldots,r_K}$ (corresponding to the tuple of coefficients $r_i \in \mathbb{Z}/l$) into homology classes. Without loss of generality, we may assume that the cycle representatives z_{α_j} corresponding to the classes α_j occurring in the decomposition themselves have size K. Letting the tuple r_i vary, one sees that for each K, one can find N and K_1 so that the sentence Φ_{K,N,K_1}

“ There exist cycles $z_1, z_2, \ldots, z_N$, all of size K, such that for any cycle z with $\|z\| = K$, either $\mathrm{dist}(z, z_1) \leqslant K_1$ or $\mathrm{dist}(z, z_2) \leqslant K_1$ or ... or $\mathrm{dist}(z, z_N) \leqslant K_1$ ”

holds over k_0. But this is first-order, so by Tarski's theorem it holds in all algebraically closed fields of the same characteristic as k_0.

Note that the ‘obvious’ thing to say (that the cycles z_i are pairwise non-homologous) is not first-order. Nonetheless, for every K there exists a least $N = f(K)$ such that (for some $K_1 < \infty$) Φ_{K,N,K_1} holds. From the triangle inequality, one sees that over each algebraically closed field, $f(K)$ is the number of distinct homology classes that can be represented by cycles of size K; so this number does not change with the underlying field. A fortiori, $H_n(\mathsf{G}(k), \mathbb{Z}/l)$ is countable for every algebraically closed k of the same characteristic as k_0.

Let now $k_1 \xrightarrow{i} k_2$ be a field extension as above, and consider the induced $H_n(\mathsf{G}(k_1), \mathbb{Z}/l) \xrightarrow{i_*} H_n(\mathsf{G}(k_2), \mathbb{Z}/l)$. It is always injective, and if the sentences Φ_{K,N,K_1} hold, it is surjective too. Indeed, suppose $\alpha \in H_n(\mathsf{G}(k_2), \mathbb{Z}/l)$ was not in the image of i_*. It would have to be represented by some $z \in Z_n(\mathsf{G}(k_2))$, say, of size K. Since the inclusion i does not change chain size, this contradicts the injectivity of i_* and the fact that the same number of homology classes can be represented by cycles of size K over k_1 as over k_2. $\square$

6. The big picture

Friedlander's conjecture concerns the effect of discretization $\mathbb{C}^\delta \to \mathbb{C}$, and this paper revolves around the — set-theoretically! — equivalent discretization $\prod_{\mathsf{P}/\mathcal{U}} \overline{\mathbb{F}}_p \to \mathbb{C}$. It is natural to ask if (or in what sense) the two are compatible. That amounts to pondering the diagram

(6.1)
$$\begin{tikzcd}
H_n^{top}(B\mathsf{G}(\mathbb{C})^\delta, \mathbb{Z}/l) \arrow[rr, two heads, "i"] \arrow[d, equal] & & H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l) \arrow[d, squiggly, no head, "j"] \\
H_n(\mathsf{G}(\mathbb{C})^\delta, \mathbb{Z}/l) \arrow[r, squiggly, no head, "s"] & H_n(\prod_{\mathsf{P}/\mathcal{U}} \mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \arrow[r, two heads, "{[\iota]}"] & \prod_{\mathsf{P}/\mathcal{U}} H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)
\end{tikzcd}$$

Here P is any infinite set of primes and $\mathcal{U}$ is any non-principal ultrafilter on P. i is induced by the continuous homomorphism $\mathsf{G}(\mathbb{C})^\delta \to \mathsf{G}(\mathbb{C})$. The left-hand vertical arrow is the canonical isomorphism between discrete group homology and homology of classifying spaces. The isomorphism s is induced by a *choice* of identification of $\mathbb{C}^\delta$ with $\prod_{\mathsf{P}/\mathfrak{U}} \overline{\mathbb{F}}_p$, while $[\iota]$ is the canonical comparison map. *Choose* any isomorphism j_p, independently for each $p \neq l$, $H_*(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \approx H_*^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l)$; j is the ultraproduct of these isomorphisms, followed by the canonical identification of $\prod_{\mathsf{P}/\mathfrak{U}} H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l)$ with $H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l)$.

Friedlander proves that i is surjective and conjectures that it is an isomorphism; Lemma 3.5 proves that $[\iota]$ is surjective, and Theorem B states that $[\iota]$ is an isomorphism if and only if i is. It is *highly* non-trivial, however, that the choices can be made compatibly so that this diagram becomes commutative — even up to isomorphism only.

Analyzing Lemma 3.5, one can show that for all G, n, and (all but perhaps finitely many) l there exist formulas $z_1(-), z_2(-), \ldots, z_{|H|}(-)$ in the language of rings such that for any algebraically closed field k, the $z_1(k), z_2(k), \ldots, z_{|H|}(k)$ are n-cycles in the bar complex of $\mathsf{G}(k)$ with $\mathbb{Z}/l$ coefficients; moreover, for almost all primes p, the cycles $z_1(\overline{\mathbb{F}}_p), z_2(\overline{\mathbb{F}}_p), \ldots, z_{|H|}(\overline{\mathbb{F}}_p)$ form exact representatives of $H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$. As a corollary, one has, for almost all l and p, *canonical* homomorphisms

$$H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \xrightarrow{h_p} H_n(\mathsf{G}(\mathbb{C})^\delta, \mathbb{Z}/l) \xrightarrow{i} H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l). \tag{6.2}$$

Either of h_p being an isomorphism or i being an isomorphism is equivalent to Friedlander's conjecture; that the composite $j_p = i \circ h_p$ is an isomorphism is Friedlander's theorem. (Note that it is rather unobvious whether either Quillen's or Friedlander's proof of $H_n(\mathsf{G}(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \approx H_n^{top}(B\mathsf{G}(\mathbb{C}), \mathbb{Z}/l)$ gives a preferred isomorphism between the two sides. The devil is in the passage between positive and zero characteristics, which in Quillen's proof hinges on a Brauer lift, and in Friedlander's an embedding of the Witt vectors of $\overline{\mathbb{F}}_p$ in the complexes.)

Friedlander's conjecture implies that the inclusion $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ induces an isomorphism $H_n(\mathsf{G}(\overline{\mathbb{Q}}), \mathbb{Z}/l) \xrightarrow{=} H_n(\mathsf{G}(\mathbb{C})^\delta, \mathbb{Z}/l)$, so the homology of $\mathsf{G}(\mathbb{C})^\delta$ must have cycle representatives that are algebraic over $\mathbb{Q}$. In fact, they must be $z_1(\overline{\mathbb{Q}}), z_2(\overline{\mathbb{Q}}), \ldots, z_{|H|}(\overline{\mathbb{Q}})$. The end result is that, under Friedlander's conjecture, (6.1) becomes *strictly* commutative for any choice of P, $\mathfrak{U}$, and s if the isomorphisms j_p are chosen as in (6.2). Perhaps this is the most beautiful embodiment of the compatibility of logic with geometry.

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