Operads from the viewpoint of categorical algebra

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Abstract. We exhibit a parallel between Lawvere's algebraic theories and operads; there is a common ancestor of both notions whose syntax is described by labeled trees, dummy variables corresponding to special labels. Operads, cyclic and braided operads alike arise as group objects in the category of *compositional structures.* Laplaza's coherence theorem for "associativity not an isomorphism" is seen to correspond geometrically to the 1-skeleton of a variant of the Stasheff polytope. Several questions are raised concerning the extent of the above-mentioned parallel; applications to higher categories remain for future work.

The purpose of this note is to locate operads within the realm of classical universal algebra, using a uniform language to describe both, and to sketch how they may be used to construct structures in category theory analogous to ones in *Set* – in the sense in which MacLane monoidal categories are analogous to monoids. There are many more pieces to be fitted within this picture, such as symmetric and braided monoidal categories, "lax" functor categories, tensor products and abelianizations of structures, that will not be touched upon here; the chosen application is to Laplaza's coherence theorem.

The thrust of the first section, mainly a review, is that what sets operads apart is not their describing structures via n-ary operations and identities between composites. (That is the very essence of universal algebra, and goes back at least to the 40's, to the notion of *clone* or "closed set of operations"; see [**12**].) Rather, it is the fact that operads cannot identify or skip inputs, only permute them. Being of such limited syntax allows operadic theories to extend to enriched categories more readily than the rest of universal algebra, and makes features common to all algebraic theories – such as free models, coequalizers, tripleability and cohomology – explicitly constructible. It also lets them extend from *Set*-based structures to ones in *Cat* (small categories) in a way that non-operadic structures do not; an aspect of this is captured, perhaps, in our notion of "relaxation". The ultimate goal of any formalism of its kind would be an analogous development for n-categories – an area where presenting algebraic structures in terms of generators and relations (commutative and "pasting" diagrams, that is) has been notoriously cumbersome. Correspondingly, the operads encountered in this article have diverse combinatorial structures (categories, graphs, posets) rather than topological or graded algebraic objects as coefficients. Finally, the essay is punctuated with (perhaps too many) questions.

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1. A glance at universal algebra

Birkhoff defines a *variety*, or *equational class of (finitary) universal algebras* by starting from a set of symbols $\{\Omega_n | n \in \mathbb{N}\}$ of n-ary operations, introducing *terms* involving *variables* as a logician would, and imposing a set of identities to hold between formal composites of these function symbols; the notion of *model* is that of *interpretation* familiar from set-based model theory. In Lawvere's reformulation of universal algebra, all possible composites of operations defining an *algebraic theory* are collected as morphisms in a category whose objects are enumerated by the natural numbers, the n^{th} object being the n^{th} categorical power of the first. A model of such a theory is a product-preserving functor into the category *Set* (or some other suitable category). The role of 'dummy variables' is in fact played by the canonical projections into the product. (See [**12**] or [**2**] for a smooth transition from Birkhoff's formulation to Lawvere's.)

En route to operads, we describe Lawvere theories via trees. To quickly get through the botany:

Definition 1.1. An **undirected graph** is given by a set V of **vertices** and a subset $E \subseteq \{\{v, w\} | v, w \in V\}$ of the unordered pairs of vertices, the **edges**. A **rooted tree** is a finite, loop-free, connected graph with a distinguished vertex called **root**. A vertex other than the root is called **terminal** if there is a unique edge incident to it. By definition, the root is also terminal in the tree containing a sole vertex and no edges.

It follows that there is a unique path from each vertex to the root. (Paths are meant not to repeat any vertex; so this may also be thought of as assigning a direction to each edge.)

There is to be an ordering of the incident edges at each vertex, assumed such that the edge contained in the path to the root is the *first* in the order. (This condition is vacuous for the case of the root vertex.) The **valence** of a vertex v, denoted $|v|$, is the number of incident edges for $v =$ the root, and that number minus one otherwise.

Let Trees denote the set of (isomorphism classes of) rooted trees with local edge-ordering as above.

It follows that one can define an ordering of the vertices on $\tau \in$ Trees, by lexicographic ordering of all the possible paths originating at the root. The root is always initial in this ordering.

In the figures that follow the root is usually emphasized by a double boundary, and edge ordering at a vertex is counterclockwise.

Let $Set^{\mathbb{N}}$ be the category of graded sets (with functions levelwise as morphisms). For $X_* \in Set^{\mathbb{N}}$, think of elements of X_n as labels of n-ary operations (so elements of X_0 are the constants of the theory). Our families of structures – defined as algebras over a free structure triple – are distinguished by the syntax permitted for inputs. First, consider

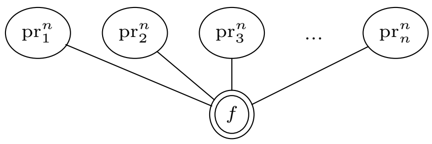
Definition 1.2. A **Lawvere tree** decorated by $X_* \in Set^{\mathbb{N}}$ and $n \in \mathbb{N}^+$ is a $\tau \in$ Trees in which to each vertex v there is assigned an element of $X_{|v|}$, *or*, but only for a terminal vertex v, one of the symbols "pr_i^n", some $1 \leq i \leq n$. (The n is to be the same for the whole tree.)

The formal symbol pr_i^n is to be thought of as the operation that takes an n-tuple and selects its i^{th} element. The *degree* of a Lawvere tree τ, denoted $\deg(\tau)$, is the n decorating it – if indeed any pr_i^n appears as label – and zero otherwise.

Define a functor $\mathcal{L} : Set^{\mathbb{N}} \to Set^{\mathbb{N}}$ as follows: let $(\mathcal{L}X)_n$ be the set of Lawvere trees decorated by X_*, of degree n. A morphism in $Set^{\mathbb{N}}$ induces a mapping between Lawvere trees by effecting the labels other than the pr_i^n.

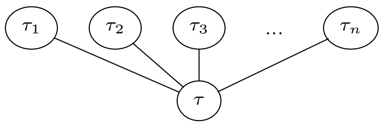
$\mathcal{L}$ is part of a monad $(\mathcal{L}, \eta, \mu)$. The natural transformation $\eta : \mathrm{Id}_X \to \mathcal{L}$ is given by sending $f \in X_n$, $n > 0$ to the tree

(1.1)



and $a \in X_0$ to a.

$\Psi \in (\mathcal{L}\mathcal{L}X)_N$ is a tree decorated by X_*-trees and "pr_i^N" (if any). Leave the latter intact. For a typical tree-vertex



(suppressing the edge – if any – leading to the root of Ψ from τ) (i) "graft in the tree τ", i.e. identify the vertex τ of Ψ with the root of τ, replacing its label as well (ii) for each occurrence of a label pr_i^n in τ replace that vertex by the root of τ_i – "graft in τ_i". (This also involves removing an edge of Ψ. For terminal vertices of Ψ, only (i) is needed.) Starting from the vertices of Ψ and recursing down to the root, one arrives at a tree $\underline{\Psi}$ decorated by X_*, i.e. an element of $\mathcal{L}X$.[1]

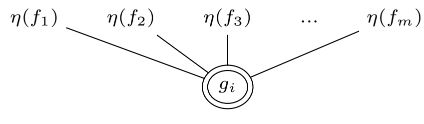
Note that no pr_i^n labels internal to some tree-vertex τ of Ψ survive to $\underline{\Psi}$, only the former's terminal "pr_i^N" (if any). So $\underline{\Psi}$ is of the same degree as Ψ, and one checks that this is a natural transformation $\mathcal{L} \xrightarrow{\mu} \mathcal{L}\mathcal{L}$.

Checking the monad identities is rather similar to the case of the free monoid on a set, save for the more tedious indexing.

Let the category of $\mathcal{L}$-algebras in $Set^{\mathbb{N}}$ be denoted by $Set^{\mathcal{L}}$. It is just the category of Lawvere theories mentioned in the introduction in disguise:

Given $(X_*, \mathcal{L}X_* \xrightarrow{\xi} X_*) \in Set^{\mathcal{L}}$, define a category $C_{\mathcal{L}}(X_*, \xi)$ as follows. Its objects are $[n]$ for $n \in \mathbb{N}$, and a morphism from $[n]$ to $[m]$ is an m-tuple $\langle f_1, f_2, \ldots, f_m \rangle$ of elements of X_n. Also add (formally) a unique morphism from $[n]$ to $[0]$.

Given a morphism $p = \langle f_1, f_2, \ldots, f_m \rangle$ from $[n]$ to $[m]$, and $q = \langle g_1, g_2, \ldots, g_k \rangle$ from $[m]$ to $[k]$, consider the tree τ_i $(i = 1, 2, \ldots, k)$



[1]It is intuitive that this leads to a unique, well-defined result. Handling operations of infinite arity is no harder; however, for trees of infinite height – i.e. transfinite computations – note that some principle of well-founded induction is needed. We will not consider such extensions of the formalism here.

where ηf_j is like the tree defined in (1.1).[2] The composite morphism $[n] \xrightarrow{p} [m] \xrightarrow{q} [k]$ is defined to be $\langle \xi(\tau_1), \xi(\tau_2), \ldots, \xi(\tau_k)\rangle$.[3]

The identity on $[n]$ is $\langle \xi(\mathrm{pr}_1^1), \xi(\mathrm{pr}_1^1), \ldots, \xi(\mathrm{pr}_1^1)\rangle$. The axioms for a category follow from (X_*, ξ) being a $\mathcal{L}$-algebra. More importantly,

Proposition 1.3. *In $C_{\mathcal{L}}(X_*, \xi)$, the object $[0]$ is terminal, and $[n]$ is the n^{th} categorical power of $[1]$, with projections $\langle \xi(\mathrm{pr}_i^n)\rangle$, $i = 1, 2, \ldots, n$.*

The proof is formal.

A morphism $(X_*, \xi) \to (Y_*, \zeta)$ in $Set^{\mathcal{L}}$ induces a functor $C_{\mathcal{L}}(X_*, \xi) \to C_{\mathcal{L}}(Y_*, \zeta)$ preserving products, and actually taking $\langle \xi(\mathrm{pr}_i^n)\rangle$ to $\langle \zeta(\mathrm{pr}_i^n)\rangle$. One can show that there is a 2-categorical equivalence between $Set^{\mathcal{L}}$ and Lawvere theories and product-preserving functors; the weak inverse requires *choosing* the preferred projections pr_i^n.

Operads. Recall definition 1.2 and add an extra condition:

Definition 1.4. An **operadic** tree is a Lawvere tree in which, if any symbol pr_i^n occurs at all, then it occurs exactly once for each $i = 1, 2, \ldots, n$.

Recall that the n (the *degree*) is to be fixed for the whole tree. A Lawvere tree of degree 0 – i.e. with no pr_i^n appearing – is tautologously operadic.

Define the functor $\mathcal{S} : Set^{\mathbb{N}} \to Set^{\mathbb{N}}$ as follows: let $(\mathcal{S}X)_n$ be the set of operadic trees decorated by X_*, of degree n. $\mathcal{S}$ is part of a monad $(\mathcal{S}, \eta, \mu)$ where the definitions of η and μ are formally identical to those for $\mathcal{L}$. (Indeed, $\mathcal{S}$ is, in an obvious sense, a *sub-monad* of $\mathcal{L}$.) Denote the category of $\mathcal{S}$-algebras in $Set^{\mathbb{N}}$ by $Set^{\mathcal{S}}$. As $\mathcal{L}$-algebras led to Lawvere's algebraic theories, so do $\mathcal{S}$-algebras to special PROP's:

Given $(X_*, \mathcal{S}X_* \xrightarrow{\xi} X_*) \in Set^{\mathcal{S}}$, define a category $C_{\mathcal{S}}(X_*, \xi)$ with objects $[n]$, $n \in \mathbb{N}$. A morphism from $[n]$ to $[m]$ is to be an m-tuple of pairs $\langle f_1, A_1, f_2, A_2, \ldots, f_m, A_m\rangle$ with $f_i \in X_{|A_i|}$ where $|A_i|$ is the cardinality of A_i. Moreover, A_i (to be thought of as the *arguments* to f_i) is a subset of $\{1, 2, \ldots, n\}$, subject to the condition that $\{A_i | i = 1, 2, \ldots, k\}$ is a partition of the set $\{1, 2, \ldots, n\}$. By definition, there is a unique morphism from $[n]$ to $[0]$.

Given a morphism $\langle f_1, A_1, f_2, A_2, \ldots, f_m, A_m\rangle$ from $[n]$ to $[m]$, and $\langle g_1, B_1, g_2, B_2, \ldots, g_k, B_k\rangle$ from $[m]$ to $[k]$, their composite is $\langle \xi(\tau_1), C_1, \xi(\tau_2), C_2, \ldots, \xi(\tau_k), C_k\rangle$ where $C_i = \bigcup_{j \in B_i} A_j$ and τ_i is defined as follows: consider the tree

$$\begin{tikzcd}[column sep=small]
f_{j_1} \arrow[drr, dash] & f_{j_2} \arrow[dr, dash] & f_{j_3} \arrow[d, dash] & \ldots & f_{j_t} \arrow[dll, dash] \\
 & & g_i & &
\end{tikzcd}$$

where $j_1 < j_2 < j_3 < \cdots < j_t$ are the elements of B_i, $g_i \in X_t$. Write w for the cardinality of C_i, and say $f_{j_l} \in X_{t_l}$. Add a top row of pr_i^w labels in increasing order

[2]This circumlocution is merely for convenience – it allowed the author to avoid typesetting a three-storey tree.

[3]The case $m = 0$ or $k = 0$ is obvious.

to make this tree operadic: pr^w_1 to $\mathrm{pr}^w_{t_1}$ go above and are connected to f_{j_1}, $\mathrm{pr}^w_{t_1+1}$ to $\mathrm{pr}^w_{t_2}$ above f_{j_2} and so on. τ_i is the resulting operadic tree.

The identity on $[n]$ is $\langle \xi(\mathrm{pr}^1_1), \{1\}, \xi(\mathrm{pr}^1_1), \{2\}, \dots, \xi(\mathrm{pr}^1_1), \{n\}\rangle$, and the axioms for a category follow as before.

In addition, define a functor $C_{\mathcal{S}}(X_*, \xi) \times C_{\mathcal{S}}(X_*, \xi) \xrightarrow{\otimes} C_{\mathcal{S}}(X_*, \xi)$ by $[n] \otimes [m] = [n+m]$ and, for a morphism $p = \langle f_1, A_1, f_2, A_2, \dots, f_N, A_N\rangle$ from $[n]$ to $[N]$ and $q = \langle g_1, B_1, g_2, B_2, \dots, g_M, B_M\rangle$ from $[m]$ to $[M]$, letting $p \otimes q = \langle f_1, A_1, f_2, A_2, \dots, f_N, A_N, g_1, n + B_1, g_2, n + B_2, \dots, g_M, n + B_M\rangle$.

Proposition 1.5. *$C_{\mathcal{S}}(X_*, \xi)$ is a tensor category with unit $[0]$, the re-association natural transformation being the identity and symmetry $[n] \otimes [m] \xrightarrow{\sigma} [m] \otimes [n]$ given by*

$$\langle \xi(\mathrm{pr}^1_1), \{n+1\}, \xi(\mathrm{pr}^1_1), \{n+2\}, \dots, \xi(\mathrm{pr}^1_1), \{n+m\},$$
$$\xi(\mathrm{pr}^1_1), \{1\}, \xi(\mathrm{pr}^1_1), \{2\}, \dots, \xi(\mathrm{pr}^1_1), \{n\}\rangle.$$

A morphism $(X_*, \xi) \to (Y_*, \zeta)$ in $Set^{\mathcal{S}}$ induces a strict monoidal functor $C_{\mathcal{S}}(X_*, \xi) \to C_{\mathcal{S}}(Y_*, \zeta)$.

One could check the MacLane pentagonal and hexagonal conditions[4] but $C_{\mathcal{S}}(X_*, \xi)$ is, at any rate, a *strict* symmetric monoidal category, and the statement amounts to checking the symmetric group action, which can be done by hand.

Compositional structures. These are May's "operads without a symmetric group action"; equivalently, they may be thought of as varieties of algebras that can be defined *without* recourse to variables – being allowed to prescribe only that certain composites of the generating operations be identically equal. The reason for singling this class out is that various n-categories are naturally algebras over them, and operads – unsurprisingly – allow a simple description in the category of compositional structures.

Recall, first of all, that since trees are connected, loop-free with a distinguished vertex and possess an ordering of edges at each vertex, there is a natural ordering of all the vertices. In particular, there is a linear ordering of the terminal vertices, some of which (in a Lawvere tree) may be decorated with a symbol pr^n_i.

Definition 1.6. An operadic tree is **compositional** if the symbols pr^n_i occur in the order $i = 1, 2, \dots, n$.

[4]In some sense, that would be putting the cart before the horse: one of the very goals of this paper is to understand, in uniform terms, how such conditions arise for algebraic constructions over *Cat* (or higher categories), and how can infinitely many commutativity conditions be enforced by a finite number of axioms. Since MacLane monoidal structures live in the cartesian closed category of small categories, where the diagrams asserted to commute by the coherence theorem do so by universal properties in the first place, "bootstrapping" is possible.

An operadic tree of degree 0 is, tautologously, compositional.

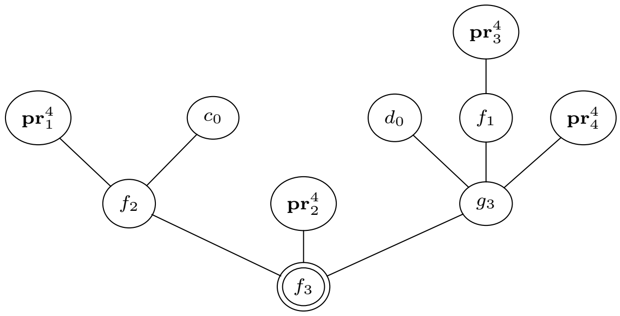
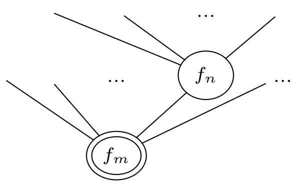


Fig.1. A compositional tree of degree 4. (Subscripts on letters indicate their valence.)

Define the functor $\mathcal{C} : Set^{\mathbb{N}} \to Set^{\mathbb{N}}$ as follows: let $(\mathcal{C}X)_n$ be the set of compositional trees decorated by X_*, of degree n. The monad $(\mathcal{S}, \eta, \mu)$ restricts to one $(\mathcal{C}, \eta, \mu)$. Denote the category of $\mathcal{C}$-algebras in $Set^{\mathbb{N}}$ by $Set^{\mathcal{C}}$. As before, one associates a category $C_{\mathcal{C}}(X_*, \xi)$ with objects $[n]$ to $(X_*, \mathcal{C}X_* \xrightarrow{\xi} X_*) \in Set^{\mathcal{C}}$; a morphism from $[n]$ to $[m]$ is an m-tuple $\langle f_1, f_2, \dots, f_m \rangle$ with $f_i \in X_{d_i}$, subject to the condition that $\sum_{i=1}^m d_i = n$. The composition of the morphism $\langle f_1, f_2, \dots, f_m \rangle$ from $[n]$ to $[m]$, and $\langle g_1, g_2, \dots, g_k \rangle$ from $[m]$ to $[k]$ is $\langle \xi(\tau_1), \xi(\tau_2), \dots, \xi(\tau_k) \rangle$ where τ_i is defined as in the operadic case, save that g_1 picks up, by default, the first $\deg(g_1)$ arguments, g_2 the next $\deg(g_2)$, and so on. Note that the labeling ends up with an unpermuted sequence of "pr_i^w", i.e. a compositional tree.

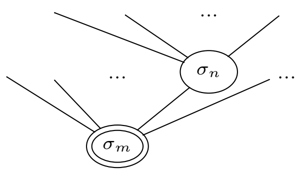
Proposition 1.7. *With $\otimes$ defined by $[n] \otimes [m] = [n+m]$ on objects and acting by juxtaposition on morphisms, $C_{\mathcal{C}}(X_*, \xi)$ becomes a strict monoidal category, and $Set^{\mathcal{C}}$-morphisms induce strict monoidal functors.*

The group object Σ. To specify an element of $Set^{\mathcal{C}}$, it suffices to give the value of the composition on two-stage trees



(the omitted labels are pr_i^n whose location is prescribed), subject to associativity for every tree with 3 non-pr_i^n vertices (such trees come in two shapes) and the value of pr_1^1 is to act as the identity.

Consider $\Sigma \in Set^{\mathbb{N}}$ having the symmetric group Σ_n in degree n. For $\sigma_n \in \Sigma_n$ and $\sigma_m \in \Sigma_m$, there is an obvious sense of



as a permutation of $m+n-1$ objects; note that when $n=0$ – we take Σ_0 to be the singleton – this amounts to the restriction of a permutation. The associativity and unit conditions satisfied, Σ becomes an object in $Set^{\mathcal{C}}$. Σ is also a group object in $Set^{\mathbb{N}}$ (i.e. a group levelwise) and the composition maps are equivariant, whence Σ becomes a group object in $Set^{\mathcal{C}}$.

PROPOSITION 1.8. *$Set^{\mathcal{S}}$ is isomorphic to Set^{Σ}, the category $Set^{\mathcal{C}}$-objects equipped with a Σ-action, and Σ-equivariant maps between them.*

This is instantaneous from the definitions; the Σ-action supplies (and is supplied by) the requisite shuffling of pr_i^n labels.

REMARK 1.9. Σ can also be thought of as a monad $(\mathbf{\Sigma}, \iota, \circ)$ on $Set^{\mathbb{N}}$: $Set^{\mathbb{N}} \xrightarrow{\mathbf{\Sigma}} Set^{\mathbb{N}}$ is given by $X_n \mapsto \Sigma \times X_n$, ι by the inclusion of the identities in Σ_n, and $\circ$ is induced by multiplications. The monads $\mathcal{C}$ and $\mathbf{\Sigma}$ interact well: there is a natural transformation $\mathcal{C}\mathbf{\Sigma} \xrightarrow{\ell} \mathbf{\Sigma}\mathcal{C}$ satisfying the identities for what Beck calls a *distributive law* of $\mathbf{\Sigma}$ over $\mathcal{C}$ (see [**1**] p.120). The action of ℓ is best described in words: an element of $(\mathcal{C}\mathbf{\Sigma}X)_n$ is a compositional tree of degree n whose vertices of valence k are decorated by pairs consisting of a function symbol from X_k and a permutation of k objects. Perform that permutation of valence edges (i.e. the edges save the one that goes towards the root – the root itself having *only* valence edges). This results in a tree whose pr_i^n labels have become permuted, i.e. an operadic tree of degree n decorated by X_*. But that can be identified with an element of $(\mathbf{\Sigma}\mathcal{C}X)_n$.

It is a purely formal consequence of this distributivity that $Set^{\mathcal{S}}$ can be equivalently defined as Σ-objects in $Set^{\mathcal{C}}$, as algebras over a certain composite triple with functor part $\mathbf{\Sigma}\mathcal{C}$, or as a lifting of the monad $\mathcal{C}$ into the category of $\mathbf{\Sigma}$-algebras (the lifting uses the natural transformation ℓ). One way or another, these tautologies are reflected in any definition of operads.

Cyclic operads. This important class was introduced by Getzler and Kapranov in [**4**], and has been aptly described by Voronov as "operads that cannot tell input from output". It amounts to dropping the criterion that the ordering around each vertex have the edge emerging towards the path as the *first*; also, for bookkeeping, it is best to add an extra edge to the root – *its* output.

More precisely, recall the definition of a *rooted tree* as a finite, loop-free, connected graph with a distinguished vertex. Add to each rooted tree another edge, one endpoint of which is the root and the other one a new vertex which we name the *exceptional vertex*. Also, assign an ordering to the edges around each vertex. This defines the set of *cyclic trees*. (This concept of tree will not be used after this subsection. Note, in particular, that the tree consisting of a single vertex – the root – is not cyclic.)

A *cyclic tree of order n decorated by* $X_* \in Set^{\mathbb{N}}$ is a cyclic tree whose vertices have been assigned labels as follows:

(i) The exceptional vertex receives a unique label, pr_0^n.
(ii) Exactly n vertices are assigned labels of the form "pr_i^n"; they must be terminal vertices. Each such label occurs exactly once for $i = 1, 2, \ldots, n$.
(iii) Other than those of type (i) and (ii), a vertex with k adjacent edges must be assigned an element of X_k.

One could now define an endofunctor $\mathcal{O} : Set^{\mathbb{N}} \to Set^{\mathbb{N}}$ as usual and show it to be part of a monad – the point being that a cyclic-tree-of-cyclic-trees "unpacks" to a cyclic tree. But we prefer to give a description that does not change the notion of tree and is in line with the previous points about $\mathbf{\Sigma}$.

The group object $\mathbf{\Sigma_+}$. Let $\Sigma_+ \in Set^{\mathbb{N}}$ have Σ_{n+1} – thought of as permutations of $\{0, 1, \ldots, n\}$ – in degree n. Σ_+ is a compositional algebra; the structure map $\mathcal{C}\Sigma_+ \xrightarrow{\xi} \Sigma_+$ is defined as follows: given a compositional tree τ decorated by Σ_+, adjoin a new edge to the root – that edge is to precede all the others, leaving the rest of the ordering of edges around the root intact – and label the other endpoint of the new edge (the "exceptional vertex") with the symbol pr^n_0.

Ignoring vertices labeled with "pr^n_i", any vertex where k edges meet is decorated by an element σ_k of Σ_k. Label the edges $0, 1, \ldots, (k-1)$ in their given order (0 therefore emerges towards the exceptional vertex) and perform σ_k on the names, i.e. re-label the edges.

Repeat this, locally, about each vertex. (Most edges possess two names, one "seen" from each endpoint, and they each will be subject to repermutation just once.)

There exists an ordering of vertices on the relabeled tree as follows: there is a unique directed path from the root to any vertex. This path can be identified with the string of edge-labels encountered;[5] apply lexicographic ordering to these strings.

In particular, the order of the labels $\mathrm{pr}^n_0, \mathrm{pr}^n_1, \ldots, \mathrm{pr}^n_n$ defines an element of Σ_{n+1}, and that is the value of $\xi(\tau)$.

The (levelwise) group structure of Σ_+ makes Σ_+ into a group object in $Set^{\mathcal{C}}$; that the structure maps are equivariant is clear. One can now define cyclic operads (in *Set*) to be the category of $Set^{\mathcal{C}}$-objects with Σ_+-action.

Question. What is the type of category naturally associated to a cyclic operad?

Proposition 1.10. *There exists a map of group objects in $Set^{\mathcal{C}}$, $\Sigma \to \Sigma_+$.*

It is induced by the inclusion $\{1, 2, \ldots, k\} \subset \{0, 1, 2, \ldots, k\}$ levelwise (thinking of Σ_k as permutations of the former set). Indeed, thanks to the fixed location of the 0-labeled edge, this just reduces to the definitions; the map is the inclusion of a group subobject.

Of course, this also defines Σ_+ as an operad. Σ_+ enjoys the same distributivity over $\mathcal{C}$ as Σ.

More general coefficient categories. It is easy to identify those features of *Set* that made its role possible. Starting with the easiest case, the free compositional structure functor, $\mathcal{C}$ can be written as

$$X_* \in Set^{\mathbb{N}} \longmapsto \{n \mapsto \coprod_{\substack{\tau \in \text{Compositional trees} \\ \deg(\tau) = n}} \mathcal{X}_{|v_1|} \times \mathcal{X}_{|v_2|} \times \cdots \times \mathcal{X}_{|v_k|}\} \in Set^{\mathbb{N}}$$

where v_i are the vertices of τ (recall they carry a canonical order) and $|v_i|$ is the valence of v_i and, by definition,

$$\mathcal{X}_{|v_i|} = \begin{cases} \{*\} & \text{if the vertex } v_i \text{ is labeled with a } \mathrm{pr}^n_i \\ X_{|v_i|} & \text{otherwise} \end{cases}$$

[5] The first edge is seen as issuing from the root and so on; there is no ambiguity.

for a fixed singleton $\{*\}$. (Note that, from here on, vertices of trees are not decorated by members of sets – save for the possible exceptional labels pr_i^n on terminal vertices.)

The definition makes sense, in place of $\langle Set, \times, \{*\}\rangle$, for any monoidal category $\langle \mathcal{M}, \otimes, I\rangle$ with countable coproducts (fix *some* particular functor computing the k-fold tensor product for each tree). The natural transformation $\mathcal{C}\mathcal{C} \xrightarrow{\mu} \mathcal{C}$ exists provided $\otimes$ commutes with coproducts, i.e. the canonical maps

$$\begin{aligned} \coprod (Y \otimes X_i) &\longrightarrow Y \otimes (\coprod X_i) \\ \coprod (X_i \otimes Y) &\longrightarrow (\coprod X_i) \otimes Y \end{aligned} \tag{1.2}$$

are isomorphisms; the monad multiplication for *Set*-trees provides the reindexing of terms for the iterated coproduct. Identifying the symmetric group with a co-power of the unit object, $\mathbf{\Sigma}_n = \coprod_{\sigma \in \Sigma_n} I \in \mathcal{M}$, $\mathbf{\Sigma}$ and $\mathbf{\Sigma}_+$ become group objects provided $\mathcal{M}$ is symmetric monoidal, allowing the definitions of operad and cyclic operad to be extended to $\mathcal{M}$.

The natural guess to extend $\mathcal{L}$ to a monoidal category $\langle \mathcal{M}, \otimes, I\rangle$ would also be

$$X_* \in \mathcal{M}^{\mathbb{N}} \longmapsto \{n \mapsto \coprod_{\substack{\tau \in \text{Lawvere trees} \\ \deg(\tau) = n}} \mathcal{X}_{|v_1|} \times \mathcal{X}_{|v_2|} \times \cdots \times \mathcal{X}_{|v_k|}\} \in \mathcal{M}^{\mathbb{N}}$$

where v_i are the vertices of τ in order, $|v_i|$ is the valence of v_i and, by definition,

$$\mathcal{X}_{|v_i|} = \begin{cases} I & \text{if the vertex } v_i \text{ is labeled with a } \mathrm{pr}_i^n \\ X_{|v_i|} & \text{otherwise.} \end{cases}$$

Note, however, that under μ I is to become the target of canonical maps from *any* object of $\mathcal{M}$. One does not expect this to happen unless I is a *terminal* object and $\otimes$ is in fact categorical product in $\mathcal{M}$. Under these conditions and (1.2), Lawvere structures can be defined by the above.

Coequalizers. The following observation relies, ultimately, on the fact that our structures have only *finitary* operations:[6]

PROPOSITION 1.11. *Suppose* $- \otimes -$ *commutes with filtered colimits; then* $\mathcal{C}$ *(*$\mathcal{L}$*, or the free operad functor, or the free cyclic operad functor) preserves filtered colimits.*

PROOF. Colimits in $\mathcal{M}^{\mathbb{N}}$ are computed levelwise; coproducts commute with colimits. Given the form of the free structure functors, the claim reduces to the following: for a filtered diagram $\mathcal{D}$ and functors $F_i : \mathcal{D} \to \mathcal{M}$, $i = 1, 2, \ldots, k$, the canonical map

$$\underset{\mathcal{D}}{colim}(F_1 \otimes F_2 \otimes \ldots \otimes F_k) \to (\underset{\mathcal{D}}{colim} F_1) \otimes (\underset{\mathcal{D}}{colim} F_2) \otimes \ldots \otimes (\underset{\mathcal{D}}{colim} F_k)$$

is an isomorphism. Indeed, "interpolate" between the two sides by

$$\underset{\mathcal{D}\times\mathcal{D}\times\cdots\times\mathcal{D}}{colim} F_1 \otimes F_2 \otimes \ldots \otimes F_k$$

[6]As far as the desired conclusion is concerned – that colimits exist in the category of $\mathcal{C}$-algebras or $\mathcal{L}$-algebras – weaker constraints on the structure suffice, such as the free algebra functor preserving κ-filtered colimits for *some* regular cardinal κ. All hypotheses cannot be dropped; indeed, there exist monads over complete and cocomplete categories whose category of algebras possesses *no* colimits (cf. [**2**]).

the colimit over the tensor product of all the diagrams $F_i : \mathcal{D} \to \mathcal{M}$. The canonical map

$$\underset{\mathcal{D}}{colim}(F_1 \otimes F_2 \otimes \ldots \otimes F_k) \to \underset{\mathcal{D}\times\mathcal{D}\times\cdots\times\mathcal{D}}{colim} F_1 \otimes F_2 \otimes \ldots \otimes F_k$$

induced by the diagonal $\mathcal{D} \to \mathcal{D} \times \mathcal{D} \times \cdots \times \mathcal{D}$ is an isomorphism since this diagonal is *final* (see [**11**]) in the filtered diagram $\mathcal{D} \times \mathcal{D} \times \cdots \times \mathcal{D}$, and the canonical map

$$\underset{\mathcal{D}\times\mathcal{D}\times\cdots\times\mathcal{D}}{colim} F_1 \otimes F_2 \otimes \ldots \otimes F_k \to (\underset{\mathcal{D}}{colim} F_1) \otimes (\underset{\mathcal{D}}{colim} F_2) \otimes \ldots \otimes (\underset{\mathcal{D}}{colim} F_k)$$

is an isomorphism by iterated evaluation of the colimit and the hypothesis on $\otimes$. □

Let now C be a category and T a monad on C. It is quite easy to show – see [**2**] – that if C has colimits of certain types of diagrams (e.g. filtered) and T preserves those, then C^T possesses colimits over those diagrams, and they are preserved by the forgetful functor.

A coequalizer is *reflexive* if its parallel morphisms possess a common section; hence it is the colimit of a filtered diagram. Recall, finally, the following classical theorem of Linton [**10**]:

THEOREM 1.12. *Suppose that C^T has coequalizers of reflexive pairs, and in addition suppose that C has finite (countable, all) coproducts. Then C^T has finite (countable, resp. all) colimits.*

In practice one verifies the hypothesis of (1.11) by noting that $- \otimes -$ has right adjoints; it follows that $\mathcal{M}$-structures such as $\mathcal{M}^{\mathcal{C}}$, $\mathcal{M}^{\mathcal{S}}$, $\mathcal{M}^{\mathcal{L}}$ possess the colimits that $\mathcal{M}$ does.

Enrichments. (See [**7**] or [**2**] vol.II. for general background on enriched category theory.) Suppose that $\mathcal{M}$ is *closed* (symmetric, cartesian) monoidal, therefore enriched over itself. This adds an extra layer of structure, as $\mathcal{C}, \mathcal{L}$, etc. can be made into $\mathcal{M}$-monads, and the corresponding categories of structures into $\mathcal{M}$-categories.

$C_{\mathcal{C}}(X_*, \xi)$ and $C_{\mathcal{S}}(X_*, \xi)$ (for $X_* \in \mathcal{M}^{\mathcal{C}}$ and $\mathcal{M}^{\mathcal{S}}$, resp.) become $\mathcal{M}$-tensor categories in the obvious sense. However, for $C_{\mathcal{L}}(X_*, \xi)$ to be an $\mathcal{M}$-category where "$[n]$" is the n$^{\text{th}}$ $\mathcal{M}$-category power of "[1]" some extra condition is needed, such as $\mathcal{M}$ being complete and cocomplete; see [**8**] for comparisons of ordinary and enriched limits.

The reason for not dwelling on these points at length is that in many cases an operad parametrizes an "up to homotopy" structure, and morphisms of interest between operads should allow for deformations of these structures. Such mapping spaces differ from the one above.

Models of structures. This paper is concerned with structures and how they may be described; still, a paragraph must be spent on what structures are for. Lawvere's equational theories are traditionally said to possess *models*, while operads (and monads in general) have *algebras*. Ignoring cyclic operads for the time, there are two approaches:

(i) Let the $\mathcal{M}$-category $\mathcal{C}$ be such that it is possible to associate to $X \in \text{ob}\,\mathcal{M}$ $End(X) \in \mathcal{M}^{\mathcal{L}}$ or $\mathcal{M}^{\mathcal{S}}$ or $\mathcal{M}^{\mathcal{C}}$, respectively. (The canonical examples are: in the compositional case, let $\mathcal{C}$ be a monoidal $\mathcal{M}$-category; in the operadic case, a symmetric monoidal $\mathcal{M}$-category; in the Lawvere case, an $\mathcal{M}$-category where $\mathcal{M}$-cartesian products exist.) For some $S_* \in \mathcal{M}^{\mathcal{L}}$, X becomes a model of S under an $\mathcal{M}^{\mathcal{L}}$-map $S_* \to End(X)$, and similarly for the other cases.[7] It is usually prescribed that $\mathrm{pr}^1_1)$ be sent to *the* identity.

(ii) Under slightly more restrictive conditions on $\mathcal{C}$, one may show that the functor defining S-structures is monadic, and even give an explicit form of the free S-model functor and monad multiplication. For $S_* \in \mathcal{M}^{\mathcal{C}}$, given that $\mathcal{C}$ is a *tensored* $\mathcal{M}$-category, F_S on objects of $\mathcal{C}$ is

$$X \mapsto \coprod_{n\in\mathbb{N}} S_n \otimes X^{\otimes n} \tag{1.3}$$

For $S_* \in \mathcal{M}^{\mathcal{S}}$, given that $\mathcal{C}$ is a *tensored* $\mathcal{M}$-category and $\mathcal{M} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ preserves (enriched) colimits (such is the case e.g. when $\mathcal{M}$ is complete), F_S can be written on objects

$$X \mapsto \coprod_{n\in\mathbb{N}} \operatorname*{colim}_{\Sigma_n} S_n \otimes X^{\otimes n} \tag{1.4}$$

where the action of Σ_n on $X^{\otimes n}$ is by permutation, and on S_n comes from the $\mathbf{\Sigma}$-action that is part of the definition of an operad.

When both approaches apply, they yield equivalent (not necessarily isomorphic) categories of models.[8]

For a Lawvere theory, the free T-model monad does not seem to allow "closed" formulas analogous to (1.3) or (1.4), save for special cases such as *Set* (more generally, a Grothendieck topos).

Question. It was completely understood in the 60's when a functor $F : \mathcal{C} \to \mathcal{D}$ is *monadic* that is, up to equivalence (or isomorphism) the forgetful part of a free $\mathbb{T}$-algebra adjunction. The answer was quickly extended to enriched categories; for $\mathcal{D} = Set$, one can characterize those monads intrinsically that arise from Lawvere's algebraic theories (see [**2**] vol.II.). But, for suitable $\mathcal{C}$ and $\mathcal{D}$, when is a monad operadic (or compositional)?

Returning to the case of an algebraic theory $T \in Set^{\mathcal{L}}$, it is easy to see that T (considered as a category with objects "$[n]$") is equivalent to the dual of the full subcategory of its *Set*-models whose objects are the free T-models on finite sets. In a way, the "coefficients" of T (i.e. the morphisms $T([n], [m])$) are determined by the free model monad.

Question. Are the coefficients of an operad (or compositional structure) determined by the free model monad?

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[7] Equivalently, one may think of this as associating a category of the type of $C_{\mathcal{L}}$, $C_{\mathcal{S}}$, or $C_{\mathcal{C}}$ to $X \in \mathrm{ob}\,\mathcal{M}$ – functorially, presumably – and the structure map exists as a functor between such categories. E.g. classically a model of a Lawvere theory T, thought of as a category whose "$[n]$" object is the n^{th} power of "[1]", is a product-preserving functor into a category $\mathcal{C}$ (usually, *Set*).

[8] Following Smirnov, [**3**] defines operads as monoids in a certain monoidal category of "$\mathcal{S}$-objects"; correspondingly, [**14**] defines algebras for an operad, in slightly more generality, as objects in a category equipped with a suitable monoidal action. These definitions are monadic and all coincide iff free monoid objects exists iff an expression like (1.4) makes sense. Whichever way, it is possible to enrich the category of models over $\mathcal{M}$ as well, and to construct colimits in a manner analogous to the case of categories of structures. For algebraic categories over *Set* – such as monoids or groups – there are well-known explicit constructions of coequalizers as equivalence classes of terms. For examples involving operads, see [**3**] or [**14**].

In view of the provocative analogy between formal power series and (1.3), and divided power series and (1.4), the question may be phrased: can one recover – up to uniqueness of the suitable kind – the Taylor expansion of a functor, if it exists?

For $\mathcal{M} = \mathcal{C} = Set$, an affirmative answer was given by Joyal in his investigation of *foncteurs analytiques* and *espècies*, cf. [**5**].

2. Relaxing presentations

This is an attempt to examine, within the context sketched above, how "strict" notions (equality, isomorphism...) mutate to "lax" ones (isomorphism, equivalence) upon passing from *Set*-based structures to ones in *Cat*. This would probably stay a fringe phenomenon in higher-order logic if the combinatorics and geometry involved did not extend its interest.

The heuristics follow. Consider some category Set^T of structures over *Set*, such as the ones encountered so far; write T for the free structure functor as well. Consider some $S_* \in Set^T$ (we will take S_* to be *monoids* as a compositional structure momentarily) and a presentation of S_* as a coequalizer $TR_* \rightrightarrows TG_* \to S_*$ in Set^T, R_* and G_* assumed finite. It is usually not the case that, on applying the forgetful functor $U : Set^T \to Set^{\mathbb{N}}$, the diagram $UTR_* \rightrightarrows UTG_* \to US_*$ is still a coequalizer in $Set^{\mathbb{N}}$.[9] It is, however, when the coequalizer $TR_* \rightrightarrows TG_* \to S_*$ is reflexive.

For such a presentation, $TR_* \rightrightarrows TG_*$ is a (reflexive) graph in each degree. Moreover, it is an object of $RGrph^T$, T-structures in the category of reflexive, directed graphs, structure maps coming from the ones on its vertices and edges. The term "relaxation" refers, quite simply, to thinking of the n-ary operation (some composite of the R_*) that used to be responsible for identifying two operations (composites of the generators G_*) only as a mere *natural transformation* or *homotopy* or *deformation* between them.

By assumption, $(UTR)_n \rightrightarrows (UTG)_n \to (US)_n$ is a coequalizer in *Set*, hence connected components of the graph $(UTR)_n \rightrightarrows (UTG)_n$ correspond to operations in S_n. One can associate what might be called a *coherent structure* to this graph: if two vertices ("operations") are linked by a directed path, connect them by a unique edge ("natural transformation"). This gives a category (poset, even) levelwise and, tautologously, an object of Cat^T, a T-structure in the category of small categories. Call it $S_*^{(1)}$. Note the property models of $S_*^{(1)}$ possess: if two n-ary functors in the structure are connected by a chain of structural natural transformations, then they are connected by a *canonical* one; the composites of canonical natural transformations are again canonical, and operations of different arities are linked by those "vertical" compositional and substitution maps that are part of the definition of any T-structure.

If $S_*^{(1)}$ permits a finite presentation in Cat^T, the procedure can be iterated: the formal properties of *Set* used above[10] are inherited by category objects in *Set*, that is, *Cat*, whence inherited by category objects in *Cat*, and so on.

Observe the following:

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[9] It is for the canonical presentation $TTS_* \rightrightarrows TS_* \to S_*$ which is *split* hence *absolute*, i.e. remains a coequalizer under the application of any functor. However, the canonical presentation just about always involves infinitely many generators and infinitely many relations.

[10] The formal properties referred to are those of any topos – even less would suffice – save that the notion of *finitely presented* requires suitable reformulation.

- Relaxation is not functorial in the structure. It is functorial in a *presentation* of the structure.
- An identity between two operations becomes relaxed into an "arrow" – a (higher) natural transformation, not necessarily assumed an isomorphism. The direction of this arrow is itself subject to choice; this fact corresponds also to the involutions existing in n-categories (see below)
- A priori, the n-fold relaxation of a *Set*-structure is a structure with coefficients in Ehresmann's n-dimensional categories, usually denoted cat^n and called cubical. More closely, however, the coefficient is a reflexive graph object in the category of reflexive graph objects in the category of reflexive ... in *Set*.

  Here, for any category C, the category of reflexive graphs in C is, by definition, the functor category $\mathrm{Fun}(\,\bullet \underset{\rightarrow}{\overset{\rightarrow}{\leftarrow}} \star\,, C)$ where the indexing diagram $\bullet \underset{t}{\overset{s}{\underset{\rightarrow}{\overset{\rightarrow}{\xleftarrow{i}}}}} \star$ is a category with two objects and three non-identity arrows, with composition $si = ti = \mathrm{id}_\star$. *RGrph* is reflexive graphs in *Set* (note that they are automatically directed), and define $RGrph^n$ to be reflexive graphs in $RGrph^{n-1}$. Objects in $RGrph^n$ form special kinds of "pasting diagrams" and may be called "spherical".
- If successive relaxations $S_*^{(0)} = S_*, S_*^{(1)}, S_*^{(2)}, S_*^{(3)}, \ldots$ exist, they give the data for a topological T-structure (e.g. operad) having a finite CW-complex in each degree; at each stage, an operation in S_* becomes replaced by a contractible CW-complex.

The following example – which is the simplest non-trivial one – may put some old objects in a new light.

Monoids, associahedra, Laplaza's coherence theorem. Consider the structure "unitless, associative monoid" as an object in $Set^{\mathcal{C}}$ (as it in fact can be described by purely compositional identities). The usual presentation – a binary pairing $(-,-)$ subject to $((-,-),-) = (-,(-,-))$ – is not reflexive, but becomes one as soon as the generator $(-,-)$ is included (tautologously) among the relations.[11] That is, consider the coequalizer in $Set^{\mathcal{C}}$

$$\mathcal{C}R_* \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{C}G_* \to M_*$$

where M_* has a singleton in every degree≥ 2; G_2 has a single pairing $(-,-)$ but $G_* \in Set^{\mathbb{N}}$ is empty otherwise; and $R_* \in Set^{\mathbb{N}}$ is non-empty in degrees 2 and 3 only, having one operation denoted $\{-,-\}$ and $[-,-,-]$ each, with $\{-,-\} \overset{s}{\longmapsto} (-,-)$, $\{-,-\} \overset{t}{\longmapsto} (-,-)$, $[-,-,-] \overset{s}{\longmapsto} ((-,-),-)$, $[-,-,-] \overset{t}{\longmapsto} (-,(-,-))$ and section $(-,-) \overset{i}{\longmapsto} \{-,-\}$.

$(\mathcal{C}R \rightrightarrows \mathcal{C}G)_* \in RGrph^{\mathcal{C}}$ since limits in functor categories are computed "pointwise". (Here and in what follows, unless mentioned otherwise, the requisite monoidal structure on a category is the cartesian one.) Denote the reflexive graph $(\mathcal{C}R \rightrightarrows \mathcal{C}G)_n$ by $\vec{K}_n$.

Recall that the vertices of the Stasheff polytope K_n are labeled by full parenthetizations of n variables (i.e. elements of $(\mathcal{C}G)_n$), and edges correspond to *single* applications of re-association. The edges may as well be thought of as directed, as

[11] In fact, any coequalizer can be changed into a reflexive one by a similar trick.

prescribed by $((-,-),-) \to (-,(-,-))$. Under this interpretation, the 1-skeleton of K_n is part of $\vec{K}_n$; here is an illustration for $n = 5$:

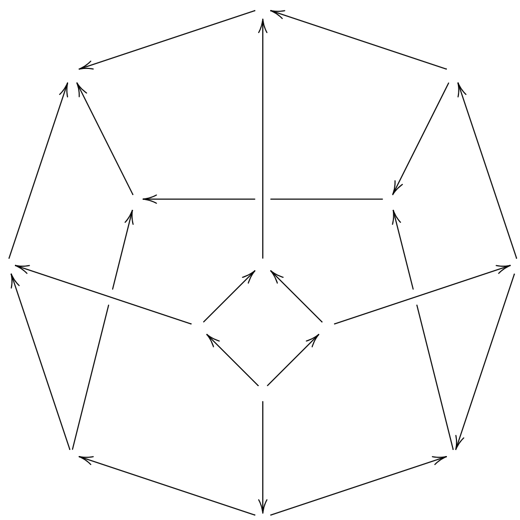


Fig.2. The Stasheff polytope K_5.

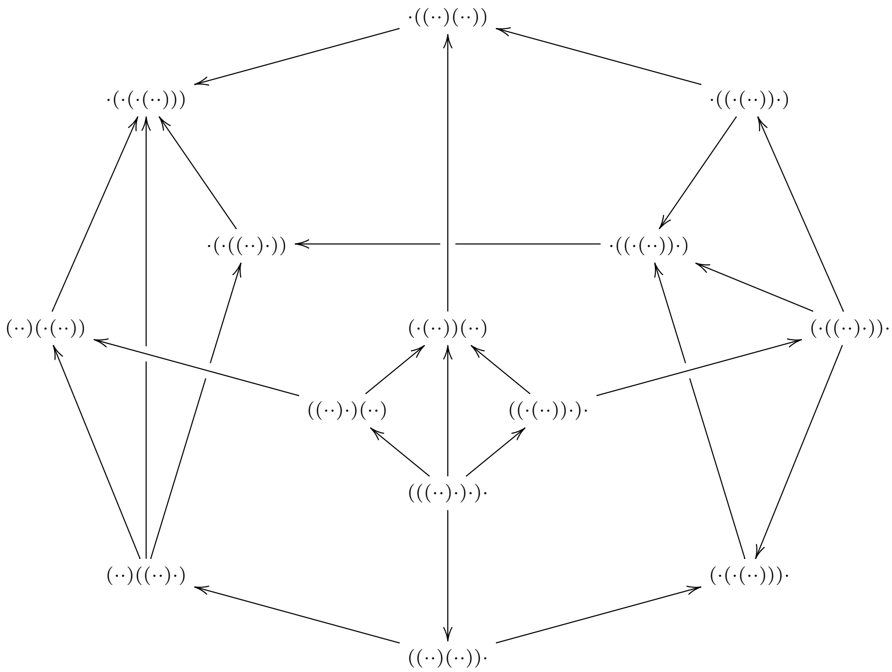


Fig.3. The graph $\vec{K}_5$. (There is also an "identity arrow" on each vertex.)
For readability, the outermost parentheses have been omitted from the labels.

Note that if the arrows are not isomorphisms, the commutativity of the pentagonal faces of K_5 does *not* imply the commutativity of the squares. The quadrilateral faces commute by functoriality: they reflect the fact that in case an expression allows more than one re-association, the order in which they are applied is irrelevant. (The three "extra" diagonals in $\vec{K}_5$ correspond to the composite symbols [[–,–,–],–,–], [–,[–,–,–],–] and [–,–,[–,–,–]], leading from ((((–,–),–),–),–) to ((–,(–,–),(–,–)) etc.) Similarly for the higher-dimensional polytopes, and the hypercubes embedded therein. Add, therefore, these commutativity conditions formally:

PROPOSITION 2.1. *The functor* $Cat^{\mathcal{C}} \to RGrph^{\mathcal{C}}$ *induced by the forgetful* $Cat \to RGrph$ *has a left adjoint* F.

F is not given by just degreewise application of the free category functor; there are constraints coming from the fact that the structure maps have to be functors. Recall therefore the notion of a (reflexive) graph with commutativity conditions (see e.g. [**2**]); it is simply $\langle G, \mathcal{P}\rangle$ where $G \in RGrph$ and $\mathcal{P}$ is a set $\{(P^0_\lambda; P^1_\lambda)|\lambda \in \Lambda\}$ of pairs of paths in G s.t. P^0_λ and P^1_λ share their initial and terminal vertices. Graphs with commutativity conditions form a category *RGrphCom*; a morphism of graphs induces a morphism of paths which is required to take a distinguished pair $\langle P^0_\lambda; P^1_\lambda\rangle$ into another distinguished pair. There is a functor $Cat \xrightarrow{U} RGrphCom$: for a category C, UC has the graph underlying C and as commutativity conditions, $(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_l)$ for every pair of composable sequences m_i and n_j in C whose composites coincide.

See [**2**] for a description of a left adjoint F' to U; intuitively, $F'G$ is the quotient of PG, the free path category on G, by the smallest equivalence relation containing the given $\mathcal{P}$.

Given $S_* \in RGrph^{\mathcal{C}}$, suppose first that $S_0 = S_1 = \emptyset$; then $FS_* \in Cat^{\mathcal{C}}$ has an easy inductive construction. Set $(FS)_0 = (FS)_1 = \emptyset$. The induction hypothesis is that $(FS)_i$ has been defined for $i < n$ as $F'\langle S_i, \mathcal{P}\rangle$ for some commutativity conditions $\mathcal{P}$ on S_i. Consider now the structure map $\mathcal{C}S_* \xrightarrow{\xi} S_*$

$$\Big(\coprod_{\substack{\tau\in\text{Compositional trees}\\ \deg(\tau)=n}} S_{|v_1|} \times S_{|v_2|} \times \cdots \times S_{|v_k|}\Big) \xrightarrow{\xi} S_n$$

where v_i are the vertices of τ in order (skipping the ones decorated by pr^n_i) and $|v_i|$ is the valence of v_i. Thanks to $S_0 = S_1 = \emptyset$, $|v_i| < n$ in the non-trivial cases, hence the left-hand side may be considered as a category:

$$\coprod_{\substack{\tau\in\text{Compositional trees}\\ \deg(\tau)=n}} (FS)_{|v_1|} \times (FS)_{|v_2|} \times \cdots \times (FS)_{|v_k|}$$

Consider now, for some tree τ, a k-tuple of morphisms $A_i \xrightarrow{m_i} B_i \in (FS)_{|v_i|}$, $i = 1, 2, \ldots, k$. By assumption, the morphism m_i can be thought of as an equivalence class of paths from A_i to B_i in $S_{|v_i|}$. Choose two paths p^0_i, p^1_i representing m_i, $i = 1, 2, \ldots, k$ such that the p^0_i have the same length for all i, and similarly for the p^1_i. Then the tuples $(p^0_1, p^0_2, \ldots, p^0_k)$ and $(p^1_1, p^1_2, \ldots, p^1_k)$ both represent $(m_1, m_2, \ldots, m_k)$ in the product category $(FS)_{|v_1|} \times (FS)_{|v_2|} \times \cdots \times (FS)_{|v_k|}$. ξ maps paths as well; add the commutativity condition $\big(\xi(p^0_1), \xi(p^0_2), \ldots, \xi(p^0_k); \xi(p^1_1), \xi(p^1_2), \ldots, \xi(p^1_k)\big)$ to S_n, for all trees and all such k-tuples of pairs. Define $(FS)_n$ as $F'S_n$ for the commutativity conditions thus obtained. Composition of morphisms in $F'S_i$ is induced by concatenation of paths, hence ξ naturally extends to a functor. Degree by degree, this defines $FS_* \in Cat^{\mathcal{C}}$.

The natural bijection $Cat^{\mathcal{C}}(FG_*, C_*) \cong RGrph^{\mathcal{C}}(G_*, UC_*)$ is easy to construct, using the same induction.

The presence of constants or non-identity unary operations ruins the "arity filtration" used above; but since it is not needed for the present application, the proof of the general case of (2.1) is postponed till the end of this section.

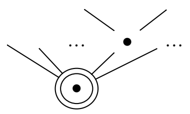
The category *Pos* of posets is reflective in the *RGrph*; the reflector $\mathit{RGrph} \xrightarrow{\mathrm{coh}} \mathit{Pos}$ is as follows: for $G \in \mathit{RGrph}$, $\mathrm{coh}(G)$ has the same vertices as G, and $A \prec B$ for two vertices A, B iff there exists a (directed) path in G from A to B. coh() is a left adjoint, but is easily seen to preserve finite limits; it follows that it induces a functor $\mathit{RGrph}^{\mathcal{C}} \to \mathit{Cat}^{\mathcal{C}}$.

Let $P_* \in \mathit{Cat}^{\mathbb{N}}$ have $\bullet \to \star$, the category with two objects and a single non-identity morphism in degree 4, and be empty otherwise. Consider the two maps $\mathcal{C}P_* \to (F\vec{K})_*$ in $\mathit{Cat}^{\mathcal{C}}$ the first of which takes $\bullet \to \star$ to the composite $(((-,-),-),-) \to ((-,(-,-)),-) \to (-,((-,-),-)) \to (-,(-,(-,-)))$ and the second, to the composite $(((-,-),-),-) \to ((-,-),(-,-)) \to (-,(-,(-,-)))$. The following is our main application:

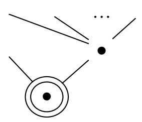
THEOREM 2.2. *The diagram* $\mathcal{C}P_* \rightrightarrows (F\vec{K})_* \to \mathrm{coh}(\vec{K}_*)$ *is a coequalizer in* $\mathit{Cat}^{\mathcal{C}}$.

This is Laplaza's "coherence for associativity not an isomorphism" theorem [**9**]: if in a compositional structure in some category monoidal over *Cat* (such as the category of small categories itself) with pairing and re-association as encoded in $\vec{K}_n = (\mathcal{C}R \rightrightarrows \mathcal{C}G)_n$, $n = 2, 3$ the single pentagonal relation P_4 is satisfied, then in fact the whole structure is coherent. The proof rests on the geometry of arrows in $(F\vec{K})_n$.

$(F\vec{K})_n$ contains $n+1$ isomorphic copies of the category $(F\vec{K})_{n-1}$, given by components of its $\mathcal{C}$-structure indexed by the compositional trees (pr_i^n labels omitted)



(the root being of valence $n-1$; there are $n-1$ trees of this type) and



and its mirror image. Call these *maximal faces.* Two maximal faces are either disjoint, or meet in the image of a product of lower-dimensional $\vec{K}_i$-cells.

Surprisingly, the fact that $\vec{K}_n$ can be realized on the $n-3$-sphere, with maximal faces spanning $n-3$-balls seems to be of limited help. Since the edges are not assumed to be isomorphisms, much depends on their orientation; also note that some diagrams in $\vec{K}_n$ commute by functoriality independently of the pentagon condition. The key lemma is:

LEMMA 2.3. *Given vertices* A, B, C *in* $\mathrm{coh}(\vec{K}_n)$ *s.t.* $A \prec B$ *and* $A \prec C$*, there exists a vertex* D *s.t.* $B \prec D$*,* $C \prec D$ *and for any vertex* X *with the property that* $B \prec X$ *and* $C \prec X$*,* $D \prec X$*.*

If all of A*,* B *and* C *lie in the same maximal face, so does* D*.*

This lemma is reminiscent of several "normal form" theorems in logic and recursion theory, saying that if an expression (i.e. A) can be changed into non-equivalent forms (B and C) by permissible moves (sequences of re-associations, i.e. an edge path), then those expressions may again be changed by permissible moves into an equivalent form, which is in fact initial or "minimal" with this property.

In the present case, the moves (i.e. shifts of parenthesis) getting from A to D is the union of the moves from A to B and from A to C; the reader is spared the details, but see [**9**].

Observe now that the coequalizer of $\mathcal{C}P_* \rightrightarrows (F\vec{K})_*$ can be computed, thanks to the absence of constants and unary operations, by degreewise induction; it is a quotient of $(F\vec{K})_*$, i.e. $F'\langle U(F\vec{K})_n, \mathcal{P}\rangle$ for some larger collection of commutativity conditions $\mathcal{P}$. Suppose that in degrees $0 \leq i < n$ the coequalizer is in fact $\mathrm{coh}(\vec{K}_i)$ (the structure maps $\mathcal{C}\,\mathrm{coh}(\vec{K}_*) \to \mathrm{coh}(\vec{K}_i)$ are then uniquely determined). The claim follows if it is shown that $\mathrm{coh}(\vec{K}_n) = F'\langle U(F\vec{K})_n, \mathcal{P}\rangle$ where the extra commutativity conditions $\mathcal{P}$ state precisely that any two parallel morphisms within the same maximal face are equal.

This follows by induction. For two vertices A, X in $(F\vec{K})_n$, denote by $\mathrm{dist}(A, X)$ the size of the longest composable chain of (non-identity) morphisms from A to X. When there is no morphism from A to X, $\mathrm{dist}(A, X)$ is undefined; otherwise, it is a finite number since $(F\vec{K})_n$ is finite and loop-free.

Given two parallel morphisms $A \overset{n}{\underset{m}{\rightrightarrows}} X$, the induction is on $\mathrm{dist}(A, X)$. Factor m, n as $A \xrightarrow{b} B \xrightarrow{m'} X, A \xrightarrow{c} C \xrightarrow{n'} X$ respectively, where b and c cannot be factored further (they therefore belong to the 1-skeleton of the Stasheff polytope K_n). By Lemma 2.3, there exists a diagram

$$\begin{tikzcd}
& B \arrow[dr] \arrow[drr, "m'"] & & \\
A \arrow[ur, "b"] \arrow[dr, "c"'] & = & D \arrow[r] & X \\
& C \arrow[ur] \arrow[urr, "n'"'] & &
\end{tikzcd}$$

It is easy to see that for $n > 4$, if there are two edges issuing from a vertex in K_n, then they both belong to the same (not necessarily unique) maximal face *or* the diagram guaranteed by Lemma (2.3) can be chosen to commute by functoriality. At any rate, the lozenge marked with $=$ commutes; observe that $\mathrm{dist}(B, X) < \mathrm{dist}(A, X)$, $\mathrm{dist}(C, X) < \mathrm{dist}(A, X)$ and use induction. Finally, for $\mathrm{dist}(A, X) = 1$, the whole diagram $A \overset{n}{\underset{m}{\rightrightarrows}} X$ must belong to one maximal face. $\square$

Proof of Prop.2.1 in the general case. Given $G_* \in RGrph^{\mathcal{C}}$, suppose that *some* commutativity conditions are added in each degree: $\{\langle G_n, \mathcal{P}_n\rangle | n \in \mathbb{N}\}$. Note that $\mathcal{C}G_* \xrightarrow{\xi} G_*$ defining G_* as a $\mathcal{C}$-algebra also induces certain maps

$$\Big(\langle G, \mathcal{P}\rangle_{|v_1|} \times \langle G, \mathcal{P}\rangle_{|v_2|} \times \cdots \times \langle G, \mathcal{P}\rangle_{|v_k|}\Big) \xrightarrow{\underline{\xi}} \langle G_n, \mathcal{P}_n\rangle$$

where the product on the left-hand side is computed in the category *RGrphCom* (given by products on the underlying graphs and pairs of tuples of commutativity conditions, as in (2.1)), by extending ξ to map paths. However, $\underline{\xi}$ may fail to be a morphism in *RGrphCom*, i.e. it may fail to map pairs of paths distinguished in the product to distinguished pairs in $\mathcal{P}_n$ – because $\mathcal{P}_n$ does not contain enough such "commutativity conditions".

Collect the $\mathcal{P}_n$ into $\mathcal{P} = \coprod_{n \in \mathbb{N}} \mathcal{P}_n$ and say that $\mathcal{P}$ is *adequate* if the induced $\underline{\xi}$ is in fact a morphism in $RGrphCom$; whence $\{\langle G_n, \mathcal{P}_n \rangle | n \in \mathbb{N}\}$ becomes an object in $RGrphCom^{\mathcal{C}}$. Observe:

- There is at least one adequate $\mathcal{P}$.
  Indeed, add *all possible* commutativity conditions to each G_n.
- If the sets $\mathcal{P}_\lambda, \lambda \in \Lambda$ are all adequate, so is $\bigcap_{\lambda \in \Lambda} \mathcal{P}_\lambda$.
  (The intersection still has the requisite closure property.)
- There is only a set of possible adequate $\mathcal{P}$.
  Indeed, all our graphs are assumed small.

Let now $\mathcal{P}^{\min}$ be the intersection of *all* adequate sets of commutativity conditions. Define $(FG)_n$ to be $F'G_n$ (recall F' is the left adjoint to the forgetful $Cat \xrightarrow{U} RGrphCom$). F' also preserves finite limits, so $\langle G_*, \mathcal{P}_*^{\min} \rangle$ that was by construction a compositional structure in $RGrphCom$ gives rise to $(FG)_* \in Cat^{\mathcal{C}}$. Adjointness follows by the choice of $\mathcal{P}^{\min}$. $\square$

Why operads? This essay intended to present operads within a spectrum of algebraic structures, from compositional (with its extremely limited syntax) to equational varieties that encompass much of actual algebra. (But not all, by far; the theory of fields is already not equational.) Obvious applications where the $\mathbf{\Sigma}$-action plays a role, such as the permutohedron or permutoassociahedron [**6**] must be left for a future paper. But it is easy to see why the considerations of the second section do not extend, e.g. to Lawvere theories. The problem is when a defining identity is between two operations of unequal arity; for example, the axiom for the inverse $xx^{-1} = \mathbf{1}$ in groups connects a unary operation with a constant. The "recipe" given for relaxing a presentation only relaxes n-ary relations between n-ary operations, leaving others *strict*. It would be interesting, and perhaps important, to understand what (if any!) the *broadest* class of universal algebraic structures is that allows well-behaved relaxation from *Set* to *Cat* and even further, "up to homotopy".

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