SHEAFIFIABLE HOMOTOPY MODEL CATEGORIES, PART II

TIBOR BEKE

Abstract. If a Quillen model category is defined via a suitable right adjoint over a sheafifiable homotopy model category (in the sense of part I of this paper), it is sheafifiable as well; that is, it gives rise to a functor from the category of topoi and geometric morphisms to Quillen model categories and Quillen adjunctions. This is chiefly useful in dealing with homotopy theories of algebraic structures defined over diagrams of fixed shape, and unifies a large number of examples.

Introduction

The motivation for this research was the following question of M. Hopkins: does the forgetful (i.e. underlying "set") functor from sheaves of simplicial abelian groups to simplicial sheaves *create* a Quillen model structure on sheaves of simplicial abelian groups? (*Creates* means here that the weak equivalences and fibrations are preserved and reflected by the forgetful functor.) The answer is yes, even if the site does not have enough points. This Quillen model structure can be thought of, to some extent, as a replacement for the one on chain complexes in an abelian category with enough projectives where fibrations are the epis. (Cf. Quillen [39]. Note that the category of abelian group objects in a topos may fail to have non-trivial projectives.) Of course, (bounded or unbounded) chain complexes in any Grothendieck abelian category possess many Quillen model structures — see Hovey [28] for an extensive discussion — but this paper is concerned with an argument that extends to arbitrary universal algebras (more precisely, finite limit definable structures) besides abelian groups. (The case of sheaves of simplicial groups was treated as early as 1984 by Gillet and Soulé in a preprint that has been published only recently [20].)

The problem we face is precisely that considered in the first part of this paper: how does one pass from a homotopy theory of algebraic structures to a homotopy theory of sheaves thereof? In the prequel it was suggested that a more comprehensive answer can be obtained at the price of employing a syntactic calculus stronger than coherent logic to specify the cofibrations. The goal of the first section is to introduce this calculus. The second section of this note proves the theorem stated in the abstract, while the third lists examples from "nature": that is, algebraic homotopy theories from the literature to which the main theorem applies. In the last section, it is shown that for sheafifiable homotopy model theories one can perform localization along a geometric morphism, and use presheaves to model the homotopy theory of sheaves. These properties were first discovered by Goerss–Jardine [21] resp. Jardine [31] for simplicial objects.

1991 *Mathematics Subject Classification.* Primary 18G55, 55U35; Secondary 18B25.

Familiarity with the first two sections of part I of this paper is helpful. Its notational and terminological conventions are retained.

1. Definable functors

The picturesque road to functors defined in terms of limits and colimits leads through *sketches*. (See Adámek–Rosický [1], Borceux [8] vol.II., or Barr–Wells [2] for detailed treatments.) Recall that a sketch is a diagram $\mathcal{D}$ together with a set U of cones and V of cocones in $\mathcal{D}$. That is, U is a set of functors $\{\mathcal{U}_\lambda^+ \to \mathcal{D} \mid \lambda \in \Lambda\}$ where $\mathcal{U}_\lambda^+$ is the categorical cone on a small diagram $\mathcal{U}_\lambda$; dually for V. A *model* of a sketch $\mathsf{S} := (\mathcal{D}, U, V)$ in a category $\mathcal{E}$ is a functor $\mathcal{D} \to \mathcal{E}$ that takes elements of U to limiting cones, and elements of V to colimiting cocones. This defines $\mathsf{S}(\mathcal{E})$, the category of S-structures in $\mathcal{E}$, as a full subcategory of the functor category $\mathcal{E}^{\mathcal{D}}$.

$\mathsf{S} = (\mathcal{D}, U, V)$ is called a *coherent* (or *geometric*) sketch if each $\mathcal{U}_\lambda$ has finitely many arrows. If, in addition, V is empty, S is a *finite limit* sketch. The *size* of S is the cardinality of the disjoint union of all arrows contained in $\mathcal{D}$, U and V.

A morphism of sketches $(\mathcal{D}_1, U_1, V_1) \xrightarrow{m} (\mathcal{D}_2, U_2, V_2)$ is a functor $\mathcal{D}_1 \to \mathcal{D}_2$ composition with which maps elements of U_1 into U_2, V_1 into V_2. It induces a functor $\mathsf{S}_2(\mathcal{E}) \to \mathsf{S}_1(\mathcal{E})$.

Definition 1.1. A sketch morphism $\mathsf{S}_1 \to \mathsf{S}_2$ is *rigid* if for any topos $\mathcal{E}$, the induced functor $\mathsf{S}_2(\mathcal{E}) \to \mathsf{S}_1(\mathcal{E})$ is an equivalence.

Example 1.2. Let $\mathcal{D}_1$ be the category with a single object $\bullet$ and its identity morphism, U_1 and V_1 empty. Let $\mathcal{D}_2$ be the category

$$\begin{tikzcd} \bullet & \bigstar \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

let U_2 contain only the cone

$$\begin{tikzcd} \bullet & \\ & \bigstar \arrow[ul] \arrow[dl] \\ \bullet & \end{tikzcd}$$

with the obvious "folding" functor into $\mathcal{D}_2$, and let V_2 be empty. The inclusion of $\bullet$ into $\mathcal{D}_2$ is a rigid sketch morphism. This is just the categorical truism that up to canonical isomorphism, any object has one cartesian square.

Example 1.3. Let $\mathcal{D}_1$ be the discrete category on two objects, $\{1\}$ and $\{2\}$. U_1 and V_1 are empty. Let $\mathcal{D}_2$ be the commutative square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \{1\} \arrow[d] \\ \{2\} \arrow[r] & \bigstar \end{tikzcd}$$

which is actually the cone on $\{2\} \to \bigstar \leftarrow \{1\}$. Let U_2 be the identity functor on $\mathcal{D}_2$, and let V_2 contain two cocones: the object $\bullet$ (thought of as a cocone on the empty diagram) and $\{2\} \to \bigstar \leftarrow \{1\}$. What this says is that $\bigstar$ is the coproduct of $\{1\}$ and $\{2\}$, $\bullet$ is the "intersection" i.e. pullback of the summands, and $\bullet$ is simultaneously an initial object.

The inclusion $\mathcal{D}_1 \hookrightarrow \mathcal{D}_2$ is a rigid sketch map. That is to say, binary coproducts are disjoint. This is not at all a tautology of limits and colimits, but is true in a topos.

Definition 1.4. A *definable functor* F from a sketch S_1 to a sketch S_2 is given by a sketch G and sketch morphisms $\mathsf{S}_1 \xrightarrow{s} \mathsf{G}$, $\mathsf{S}_2 \xrightarrow{t} \mathsf{G}$ such that s is rigid. It is said to be *coherently*, *finite limit* resp. *countably* defined if its graph G is such.

For any topos $\mathcal{E}$, a definable functor induces an actual functor $\mathsf{S}_1(\mathcal{E}) \xrightarrow{F_{\mathcal{E}}} \mathsf{S}_2(\mathcal{E})$ as the composite $\mathsf{S}_1(\mathcal{E}) \xrightarrow{s^{-1}} \mathsf{G}(\mathcal{E}) \xrightarrow{t} \mathsf{S}_2(\mathcal{E})$. (The indeterminacy of the quasi-inverse s^{-1} is precisely the indeterminacy of objects with universal properties, which we assume solved by choosing, once and for all, functorial limits and colimits.)

Example 1.5. The nerve functor, from category objects to simplicial diagrams in $\mathcal{E}$, is finite limit definable (for any category $\mathcal{E}$ with pullbacks, in fact). Indeed, the notion of category is definable by a finite limit sketch (the diagram that underlies it is the familiar truncated simplicial object) and for G take the simplicial indexing category Δ^{op} together with all the limit cones it contains (they are iterated pullbacks).

Example 1.6. The barycentric subdivisions of the affine simplices combine to give a functor $\Delta \xrightarrow{\mathrm{sd}} SSet$. Recall that Kan's simplicial extension functor $SSet \xrightarrow{\mathsf{Ex}} SSet$ sends $X \in SSet$ to the presheaf $\hom_{SSet}(\mathrm{sd}(-), X)$. Expressing $\mathrm{sd}(\underline{n})$, $\underline{n} \in \Delta$, as a finite colimit of representables, one obtains a definition of Ex in terms of finite limits, now valid for any category of the form $\mathcal{E}^{\Delta^{\mathrm{op}}}$. An analogous argument shows that any right adjoint between presheaf topoi is definable in terms of limits.

Example 1.7. Take a finitary single-sorted equational theory of universal algebras, and let T be the functor $Set \to Set$ underlying the free algebra functor. Let $\mathcal{D}$ be the (countable) diagram with objects pairs (X, α) where X is a finite ordinal and $\alpha \in T(X)$. An arrow $(X, \alpha) \to (Y, \beta)$ is a function $X \xrightarrow{f} Y$ such that $T(f)(\alpha) = \beta$. Let U be an object of a topos $\mathcal{E}$. Consider the functor $\mathcal{D}^{\mathrm{op}} \xrightarrow{F} \mathcal{E}$ that takes (X, α) to $U^{|X|}$ (U to the categorical power of the cardinality of X) and where $F(f)$ is induced by the projections. colim F is the free T-algebra on U. (Thinking of universal algebras, after Lawvere, as functors, this comes from the canonical presentation of a presheaf as colimit of representables.) So, the functor $\mathcal{E} \to \mathcal{E}$ taking an object to the one underlying the free T-algebra on it is countably, coherently definable. With some more work, the structure maps, hence the free algebra functor $\mathcal{E} \to \mathcal{E}^{\mathbb{T}}$ is definable as well.

The homotopically minded reader is encouraged to skim Lemma 1.9 and Prop. 1.10, then proceed to section 2.

Coherently definable functors between coherent theories enjoy an equivalent, beautifully simple definition using classifying topoi. Let $\mathbb{B}[T]$ denote the classifying topos of the coherent theory[1] T. A sketch morphism $T_1 \to T_2$ is rigid iff the induced geometric morphism $\mathbb{B}[T_2] \to \mathbb{B}[T_1]$ is an equivalence. A definable functor from T_1-models to T_2-models is a model of T_2 in $\mathbb{B}[T_1]$, that is, geometric morphism $\mathbb{B}[T_1] \xrightarrow{F} \mathbb{B}[T_2]$. The effect of F on a T_1-model in a topos $\mathcal{E}$, i.e. topos morphism $\mathcal{E} \to \mathbb{B}[T_1]$, is composition with F. The classifying topos, by its very construction, subsumes the intermediate step of enlarging the language of T_1 by coherent definitions — which was G, the "graph" of the functor, as sketched above.

Recall that the classifying topos of a finite limit theory T is a presheaf topos $\mathrm{Pre}(\mathcal{C}_T)$, where $\mathcal{C}_T$ is a small category with finite limits: $\mathrm{fpMod}_T(\mathit{Set})^{\mathrm{op}}$, the opposite of the category of finitely presentable T-models in Set. T is countable iff $\mathcal{C}_T$ is. A finite limit definable functor between finite limit theories T_1, T_2 gives rise to a finite limit preserving functor $\mathcal{C}_{T_1} \to \mathcal{C}_{T_2}$ between categories with finite limits, namely $\mathrm{fpMod}_{T_1}(\mathit{Set})^{\mathrm{op}} \to \mathrm{fpMod}_{T_2}(\mathit{Set})^{\mathrm{op}}$. It is classified by an essential geometric morphism $\mathrm{Pre}(\mathcal{C}_{T_1}) \xrightarrow{f} \mathrm{Pre}(\mathcal{C}_{T_2})$ with a finite limit preserving far left adjoint $f_!$; that is, "two geometric morphisms in one": $f^* \dashv f_*$ and $f_! \dashv f^*$.

Remark 1.8. The (co)unit natural transformations for $f^* \dashv f_*$ show that this pair of geometric morphisms provides, in the terminology of Joyal–Wraith [34], a natural homotopy equivalence between $\mathrm{Pre}(\mathcal{C}_{T_1})$ and $\mathrm{Pre}(\mathcal{C}_{T_2})$. More generally, any coherently definable adjunction between coherent theories yields a natural homotopy equivalence between classifying topoi.

Lemma 1.9. *Let S_1, S_2 be finite limit sketches, R a finite limit definable functor from S_1-structures to S_2-structures. For any topos $\mathcal{E}$, $\mathsf{S}_1(\mathcal{E}) \xrightarrow{R_{\mathcal{E}}} \mathsf{S}_2(\mathcal{E})$ preserves filtered colimits and all limits.*

The statement is equivalent to the following: if $\mathsf{S}_1 \xrightarrow{m} \mathsf{S}_2$ is a morphism of finite limit sketches, then the induced functor $\mathsf{S}_2(\mathcal{E}) \to \mathsf{S}_1(\mathcal{E})$ preserves filtered colimits and finite limits, for any topos $\mathcal{E}$. Now the diagram

$$\begin{tikzcd} \mathcal{E}^{\mathcal{D}_2} \arrow[r, "m^*"] & \mathcal{E}^{\mathcal{D}_1} \\ \mathsf{S}_2(\mathcal{E}) \arrow[r] \arrow[u, hook] & \mathsf{S}_1(\mathcal{E}) \arrow[u, hook] \end{tikzcd}$$

(by definition of the bottom functor) commutes. The vertical arrows are inclusions of full, reflexive subcategories that preserve and reflect all limits and filtered colimits. (Note that finite limits commute with filtered colimits in a topos.) The top horizontal arrow preserves all limits and colimits. $\square$

Proposition 1.10. *Let S_1, S_2, R be as above. There exists a coherently definable functor L from S_2-structures to S_1-structures such that for any topos $\mathcal{E}$,*

$$\mathsf{S}_1(\mathcal{E}) \underset{R_{\mathcal{E}}}{\overset{L_{\mathcal{E}}}{\leftrightarrows}} \mathsf{S}_2(\mathcal{E})$$

is an adjunction.

[1]In the context of the classifying topos, we use the term "theory" interchangeably with "sketch".

Proof. R is the direct image of a topos morphism; let L be the inverse image. Note that L is the *direct* image of a geometric morphism as well. The (co)unit natural transformations for $L \dashv R$

$$\begin{tikzcd}
\mathbb{B}[\mathsf{S}_1] \arrow[rr, dash, "\mathsf{Id}"] \arrow[dr, "R"'] & & \mathbb{B}[\mathsf{S}_1] \\
& \mathbb{B}[\mathsf{S}_2] \arrow[ur, "L"'] \arrow[u, phantom, "\Uparrow"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{B}[\mathsf{S}_1] \arrow[dr, "R"] & \\
\mathbb{B}[\mathsf{S}_2] \arrow[ur, "L"] \arrow[rr, dash, "\mathsf{Id}"'] & \arrow[u, phantom, "\Uparrow"] & \mathbb{B}[\mathsf{S}_2]
\end{tikzcd}$$

are the universal examples of the adjunction maps. That is to say, fix any topos $\mathcal{E}$, and let $X \in \mathsf{S}_2(\mathcal{E})$, $Y \in \mathsf{S}_1(\mathcal{E})$. A morphism $\mathsf{S}_1(LX, Y)$ is represented by a natural transformation from $\mathcal{E} \xrightarrow{X} \mathbb{B}[\mathsf{S}_2] \xrightarrow{L} \mathbb{B}[\mathsf{S}_1]$ to $\mathcal{E} \xrightarrow{Y} \mathbb{B}[\mathsf{S}_1]$. The composite natural transformation

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "Y"] \arrow[dr, "X"'] & & \mathbb{B}[\mathsf{S}_1] \arrow[dr, "R"] & \\
& \mathbb{B}[\mathsf{S}_2] \arrow[ur, "L"'] \arrow[u, phantom, "\Uparrow"] \arrow[rr, dash, "\mathsf{Id}"'] & \arrow[u, phantom, "\Uparrow"] & \mathbb{B}[\mathsf{S}_2]
\end{tikzcd}$$

represents the adjoint map in $\mathsf{S}_2(X, RY)$. The reverse direction is given by pasting in the other triangle; that they are inverse bijections follows from the (co)unit identities for $L \dashv R$. □

Remark 1.11. The deceptive simplicity of the argument is due to the presence of classifying topoi. For a typical left adjoint of the type above — say, the free universal algebra functor or the colimit of A_∞-algebras — the "recipe" one gets from the proof is quite ineffective. Nonetheless, one sees that if S_1, S_2 and R were countable, L can be countably defined as well.

Remark 1.12. Giraud's theorem, in effect, provides an axiomatization of all non-trivial "interchange properties" of *finite* limits and arbitrary colimits, that is, exactness properties of a Grothendieck topos. Makkai's [35] provability formalism and completeness theorem for sketches allows for a syntactic generation of all rigid morphisms, and so a constructive approach to definable functors, for other semantics as well. ([4] spells out the case of functors definable on diagram categories, that is, sketches with no cones and cocones, interpreted in an arbitrary category.)

2. Creating Quillen model structures via right adjoints

For $\mathcal{C} \xrightarrow{F} \mathcal{D}$ a functor and U a collection of morphisms of D (of $\mathcal{C}$, resp.), write $F^{-1}(U)$ for $\{m \in \mathcal{C} \mid F(m) \in U\}$; resp. $F(U)$ for $\{F(m) \mid m \in U\}$.

Definition 2.1. Let $\mathcal{M}$, with data $\mathsf{cof}; \mathsf{W}; \mathsf{fib}$ be a homotopy model category and $\mathcal{C} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{M}$ an adjunction. If LLP; $R^{-1}(\mathsf{W})$; $R^{-1}(\mathsf{fib})$ give a Quillen model structure on $\mathcal{C}$, say that model structure is *created by* R from the one on $\mathcal{M}$. (Here "LLP" is an abbreviation for the class of morphisms having the left lifting property w.r.t. every acyclic fibration, these latter being $R^{-1}(\mathsf{W}) \cap R^{-1}(\mathsf{fib})$.)

Although this kind of situation is as old as Quillen model categories [39], and recurrent throughout their study, it doesn't seem to have earned its own name yet. (I owe the nomenclature used above to M. Hopkins.) The next proposition gives a sufficient condition for creation to occur. It has cognates in a great number of papers, e.g. Blanc [7], Cabello–Garzón [11], Crans [13], Goerss–Jardine [22], Dwyer–Hirschhorn–Kan [15], Quillen [39], Rezk [40], Spaliński [41]. Recall

Definition 2.2. Let $\mathcal{C}$ be a cocomplete category, I any class of morphisms of $\mathcal{C}$.

- Close the class of all pushouts of I under transfinite composition in $\mathcal{C}$. This defines the class $\mathsf{cell}(I)$ of *relative I-cellular maps.*
- The class $\mathsf{cof}(I)$ of I-cofibrations is defined as follows: $X \xrightarrow{c} Y \in \mathsf{cof}(I)$ iff c is a retract of an $X \xrightarrow{r} Z \in \mathsf{cell}(I)$ in the category $X/\mathcal{C}$ of objects under X.
- I-fibrations, or I-injectives, denoted $\mathsf{inj}(I)$, are the morphisms with the right lifting property w.r.t. I; that is, such that in any commutative square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r] \arrow[ur, dotted] & \bullet \end{tikzcd}$$

with $i \in I$, $p \in \mathsf{inj}(I)$, a dotted lift making both triangles commute exists.

Proposition 2.3. *Let $\mathcal{M}$ be a model category, $\mathcal{C} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{M}$ an adjunction. Suppose*

(0) *For some set I of maps in $\mathcal{M}$, $\mathsf{inj}(I)$ are precisely the acyclic fibrations, and for some set J, $\mathsf{inj}(J)$ are precisely the fibrations. (True, for example, when $\mathcal{M}$ is cofibrantly generated.)*
(1) *$\mathcal{C}$ is (co)complete, and every set of maps of $\mathcal{C}$ permits the small object argument. (This holds, for example, when $\mathcal{C}$ is a locally presentable category.)*
(2) *Weak equivalences are closed under filtered colimits in $\mathcal{M}$.*
(3) *R preserves filtered colimits.*
(4) *For any $f \in J$, and any pushout g of $L(f)$ in $\mathcal{C}$, $R(g)$ is a weak equivalence in $\mathcal{M}$.*

Then R creates a cofibrantly generated model structure on $\mathcal{C}$.

Proof. Axioms **M1**, **M2**, **M3**, and one half of **M4** are gratis. The factorizations needed for **M5** are constructed, of course, via the small object argument. Applying the small object argument to $L(I)$, we see that every morphism of $\mathcal{C}$ can be factored as fc with $c \in \mathsf{cell}(L(I))$, $f \in \mathsf{inj}(L(I))$. By adjunction, $L(i)$, $i \in I$, is a cofibration in $\mathcal{C}$; since any LLP class is closed under the operations occurring in the definition of cell, c is a cofibration in $\mathcal{C}$. Adjointly, $R(f)$ has the right lifting property w.r.t. every $i \in I$; by definition g an acyclic fibration in $\mathcal{C}$. Analogously, applying the small object argument to $L(J)$, we see that every map in $\mathcal{C}$

can be factored as gd with $d \in \mathsf{cell}(L(J)$, $g \in \mathsf{fib}(L(J))$. Since d is a transfinite composition of pushouts of $L(j)$, $j \in J$, (2), (3), (4) and transfinite induction imply that $R(d)$ is a weak equivalence in $\mathcal{M}$. $R(g)$ has the right lifting property w.r.t. every $j \in J$, so g is a fibration in $\mathcal{C}$. By an argument similar to the case of c, d is a $\mathcal{C}$-cofibration. The missing half of **M4** follows by the *retract argument*: given an acyclic cofibration d of $\mathcal{C}$, factor it as he with h a $\mathcal{C}$-fibration, $e \in \mathsf{cell}(L(J))$. By **M2**, h is an acyclic fibration. Hence the composite he has the left lifting property w.r.t. h, which works out to mean that d is a retract of e. Since e had the left lifting property w.r.t. all fibrations, so does d. □

Assumptions (2) and (3) are not necessary for the conclusion, but seem to be satisfied in practice. The next three remarks elaborate these and a related point.

• Property (2) does not seem to follow from Quillen's axioms. If $\mathcal{M}$ is a simplicial model category, i.e. is enriched over *SSet* with good interaction between the homotopy model structures on *SSet* and $\mathcal{M}$, then something stronger than (2) holds, viz. the weak equivalences are closed under filtered colimits in the category of morphisms of $\mathcal{M}$. The same holds for other "good" enrichments, and (for a different reason) for any coherently definable homotopy model structure; see [5].

• In all *algebraic* situations I am aware of, creation happens across a finitary adjunction, i.e. one where the right adjoint preserves filtered colimits, and this is the condition easy to check. What one exploits in the proof is only that R preserves transfinite compositions of weak equivalences.

• (4) is necessary for the conclusion: if $\mathcal{C} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{M}$ is to create a model structure, L must preserve acyclic cofibrations, acyclic cofibrations are preserved by pushouts, and R is to preserve weak equivalences. But this does not make (4) easy to prove a priori.[2] Instead, it has the curious advantage that *given* it holds for a definable adjunction between *Set*-based structures, *then* it holds for the analogous adjunction between structured sheaves. The present note (and its predecessor) are concerned precisely with this relative situation.

Theorem 2.4. *Let* S_1 *and* S_2 *be finite limit structures,* R *a functor from* S_2*-structures to* S_1*-structures defined in terms of finite limits. Let* W *and* C *be sets of coherent axioms in the language of morphisms of* S_1*-structures. Let* $\mathsf{W}(\mathcal{E}) := \{f \in \operatorname{mor} \mathsf{S}_1(\mathcal{E}) \mid f \models W\}$, $\mathsf{C}(\mathcal{E}) := \{f \in \operatorname{mor} \mathsf{S}_1(\mathcal{E}) \mid f \models C\}$. *Suppose*

- *all the syntactic ingredients —* S_1, S_2, R, W, C *— are countable*

[2]Indeed, even for the "degenerate" case of $\mathcal{C} = \mathcal{M} = SSet$, L, R the identity, I am aware of no elementary proof of the fact that a pushout of an acyclic cofibration is a weak equivalence. By "elementary", I mean a proof proceeding in coherent logic; since the statement is a coherent implication valid in any topos, one knows by an abstract completeness theorem that such a proof must exist. Note that the use of geometric realization or minimal fibrations renders a proof non-elementary in this technical sense. The existence of minimal simplicial fibrations, for example, uses the axiom of choice, and need not hold in a category of sheaves.

- *for every topos* $\mathcal{E}$, $\mathsf{S}_1(\mathcal{E})$ *with weak equivalences* $\mathsf{W}(\mathcal{E})$ *and cofibrations* $\mathsf{C}(\mathcal{E})$ *is a cofibrantly generated Quillen model category*
- $\mathsf{S}_2(Set) \xrightarrow{R_{Set}} \mathsf{S}_1(Set)$ *creates a homotopy model structure on* $\mathsf{S}_2(Set)$.

Then for every topos $\mathcal{E}$, $\mathsf{S}_2(\mathcal{E}) \xrightarrow{R_{\mathcal{E}}} \mathsf{S}_1(\mathcal{E})$ *creates a Quillen model structure on* $\mathsf{S}_2(\mathcal{E})$.

Proof. Apply Prop. 2.3. (0) is an assumption. The category of models of a finite limit structure in a locally presentable category (in particular, Grothendieck topos) is locally presentable, and models of coherent axioms are closed under filtered colimits. (3) is lemma 1.9. Property (4) holds in $\mathcal{E} = Set$ by assumption, for any acyclic cofibration j in fact. But (4) is a coherent deduction: the original class of cofibrations was coherently definable; apply Prop. 1.10 to L, the left adjoint to R; that a square be a pushout of S_2-structures is coherently expressible, and so is the desired conclusion that $R(g)$ be a weak equivalence. By the theorem of Makkai–Reyes [36] that the countable fragment of coherent logic has enough models in *Set*, (4) carries over to an arbitrary topos. □

Remark 2.5. The cardinality condition in 2.4 can be bypassed with the proviso that R creates a Quillen model structure on $\mathsf{S}_2(\mathrm{Sh}(\mathcal{B}))$ for every complete Boolean algebra $\mathcal{B}$, equipped with its canonical topology.

Remark 2.6. In checking that $\mathsf{S}_2(Set) \xrightarrow{R_{Set}} \mathsf{S}_1(Set)$ creates a Quillen model structure, one is not limited to Prop. 2.3. It may be easier to verify Def. 2.1 indirectly, or to exploit special properties of the category *Set*.

The description of cofibrations in $\mathsf{S}_1(\mathcal{E})$ is glaringly non-constructive; this can be amended somewhat.

Lemma 2.7. *Suppose* $\mathcal{C} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{M}$ *creates a homotopy model structure on* $\mathcal{C}$, *any set of maps in* $\mathcal{C}$ *permits the small object argument, and* $\mathcal{M}$ *is cofibrantly generated. Let* I *be any collection (possibly proper class) of cofibrations in* $\mathcal{M}$ *that includes a generating set. Then the cofibrations created by* R *are* $\mathsf{cof}(L(I))$.

Indeed, L will take cofibrations to cofibrations, which are closed under the operations making up $\mathsf{cof}(-)$, so $\mathsf{cof}(L(I))$ is a subclass of $\mathcal{C}$-cofibrations. But for any generating set I_g of $\mathcal{M}$-cofibrations, $\mathsf{cof}(L(I_g))$ already includes all $\mathcal{C}$-cofibrations by adjunction and the small object argument. □

Corollary 2.8. *In Thm. 2.4, cofibrations in* $\mathsf{S}_2(\mathcal{E})$ *are* $\mathsf{cof}(L_{\mathcal{E}}(\mathsf{C}(\mathcal{E})))$.

Corollary 2.9. *2.4 gives rise to a functor* $\mathsf{TOPOI} \to \mathsf{HOMODEL}$ *taking* $\mathcal{E}$ *to the model category* $\mathsf{S}_2(\mathcal{E})$ *with weak equivalences* $R^{-1}\mathsf{W}(\mathcal{E})$, *cofibrations* $\mathsf{cof}(L(\mathsf{C}(\mathcal{E})))$.

Just observe that a topos morphism will induce an adjunction between the category of models, and the inverse image functor preserves weak equivalences and cofibrations. □

3. Examples

Set S_1 :=simplicial objects, with cofibrations the monomorphisms and weak equivalences defined "stalkwise". It is classical that this satisfies the conditions of Thm. 2.4; see e.g. part I of this paper for details. Many examples of creation from $\mathcal{E}^{\Delta^{\mathrm{op}}}$ have been discovered. Each entry has the following format: the adjunction $\mathsf{S}_2(Set) \leftrightarrows \mathsf{S}_1(Set)$ that defines the homotopy theory; references to the literature, where applicable; and additional remarks and questions. Checking that the conditions of 2.4 are satisfied is usually simple, and left to the reader.

Example 3.1. bisimplicial sets: $BiSSet \overset{L}{\underset{D}{\leftrightarrows}} SSet$

D is the functor of restriction to the diagonal. The left adjoint L is a left Kan extension. The theorem that D creates a model structure on $BiSSet$ is due to Moerdijk [37]. It was extended to bisimplicial objects in a topos by Crans [13]. Note that the methods — if not the words — of Thm. 2.4 are all contained in Crans' paper.

Remark 3.2. The adjunction above is in fact a Quillen equivalence, since the unit and counit maps are weak equivalences. This fact (which is stronger than what is needed to ensure that a Quillen pair induce an equivalence on the homotopy category) also survives from Set to sheaves.

Example 3.3. cyclic sets: $\mathrm{Pre}(\Lambda) \overset{L}{\underset{i^*}{\leftrightarrows}} SSet$

Λ is Connes' indexing category of cyclic sets, with a canonical inclusion $\Delta \xrightarrow{i} \Lambda$. The left adjoint L to the forgetful functor i^* from cyclic to simplicial sets is again a left Kan extension. Dwyer–Hopkins–Kan [18] prove that i^* creates a Quillen model structure on cyclic sets. Its cofibrations enjoy a combinatorial description (a rare exception!); see [18] or [5].

Remark 3.4. The preceding two examples represent an even narrower type of "simplicial creation", namely, when the adjunction is one of the form $f_! : \mathrm{Pre}(\mathcal{D}) \leftrightarrows SSet : f^*$ induced by a functor $\mathcal{D} \xrightarrow{f} \Delta$. In such a case, for f^* (i.e. precomposition by f^{op}) to create a homotopy model structure it is sufficient and necessary that $f^*(f_!(i_n^k))$ be a weak equivalence in $SSet$ for each of the horn-inclusions (or "generating acyclic cofibrations") i_n^k. To that end, as pointed out by Dwyer–Hopkins–Kan [18], it is sufficient that $\mathcal{D} \xrightarrow{y} \mathrm{Pre}(\mathcal{D}) \xrightarrow{f_!} SSet$ (where y is the Yoneda functor) be a diagram of weak equivalences in $SSet$. It would be interesting to have a criterion directly in terms of f.

Example 3.5. small categories: $Cat \xrightarrow{\mathsf{nerve}} SSet \xrightarrow{\mathsf{Ex}^2} SSet$

The right adjoint is the composite displayed above (Ex^2 being the double iteration of Kan's simplicial extension functor). The left adjoint is $C \circ \mathrm{Sd}^2$, where Sd is Kan's simplicial subdivision functor and C is "categorification" of a simplicial set. It is due to Thomason [43] that the right adjoint creates a homotopy model structure on Cat. (Tagging on the Ex^2 at the end leaves the class of weak equivalences in Cat unchanged, but does affect the (co)fibrations.) Remark 3.2 applies here as well.

Example 3.6. simplicial groupoids: $Grpd^{\Delta^{\mathrm{op}}} \xrightarrow{\mathsf{nerve}^{\Delta^{\mathrm{op}}}} BiSSet \xrightarrow{D} SSet$

As emphasized by the notation, simplicial groupoids mean here simplicial objects in the category of groupoids (equivalently, groupoid objects in *SSet*) as opposed to the objectwise discrete simplicial groupoids of Dwyer–Kan [16]. That the above composite creates a model structure is unpublished work of I. Moerdijk and S. Crans.

Remark 3.7. Small categories, simplicial groupoids and bisimplicial sets are Quillen model categories with the following in common: they are countable structures defined by finite limits; their weak equivalences have a countable coherent definition (ditto for fibrations); and their cofibrations have the form $\mathsf{cof}(J)$, J being countably many maps from a countably, coherently defined class. It is tempting to think of these syntactic properties as being the equivalent of "combinatorial homotopy" in the world of Quillen's axioms. The above examples are actually much closer tied, each being definably Quillen-equivalent to simplicial sets. I suspect these observations extend to other (intuitively) combinatorial models of spaces, such as cubical sets, Joyal's Θ-sets or Golasiński's homotopy theory of categories [23], [24].

n-types, that is to say, n-coconnected spaces, possess combinatorial models (in both the above syntactic and the intuitive senses) as well; we turn to these next.

Example 3.8. groupoids: $\mathit{Grpd} \overset{\pi}{\underset{\mathsf{nerve}}{\leftrightarrows}} \mathit{SSet}$

The $n = 1$ case is classical and simple. π is the fundamental groupoid. The nerve functor creates a homotopy structure on the category of groupoids, the cofibrations having a very simple description: functors that are injective on the object part. This model structure was sheafified in Joyal–Tierney [32] as well.

Example 3.9. 2-groupoids: $2 - \mathit{Grpd} \overset{W}{\underset{N}{\leftrightarrows}} \mathit{SSet}$

Here a 2-groupoid is a strict 2-category with all (one and two-dimensional) morphisms strictly invertible. Taking as morphisms 2-functors preserving all the structure on the nose, they form a category $2 - \mathit{Grpd}$ which is the category of models of a finite limit theory.[3] N is a combinatorial nerve functor, W its left adjoint constructed in Moerdijk–Svensson [38]. (The direct 2-categorical definition of $f \in 2 - \mathit{Grpd}$ being a fibration, as given in [38], is in fact equivalent to $N(f)$ being a Kan fibration.) The corresponding homotopy category is that of spaces with vanishing homotopy above dimension 2. The sheafified version can also be found in Crans [13].

Remark 3.10. N (which is also described, for example, in Street [42]) has a more canonical alternative B: the composite of the iterated nerve $2 - \mathit{Grpd} \to \mathit{BiSSet}$ with the diagonal $\mathit{BiSSet} \to \mathit{SSet}$. There exists a natural transformation $B \to N$ which is in fact always an acyclic fibration. B takes combinatorial fibrations to Kan fibrations as well. Thus B, if it creates a homotopy structure too, creates one that is Quillen-equivalent to that due to N.

Example 3.11. 3-groupoids: $3 - \mathit{Grpd} \overset{\hat{t}_3}{\underset{N_3}{\leftrightarrows}} \mathit{SSet}$

The $n = 3$ case has been worked out by C. Berger [6]. A *lax 3-category* is a certain partial

[3] This would fail if one chose as morphisms the "weak homomorphisms" of 2-groupoids, namely functors that preserve composition of 1-arrows only up to a (coherently chosen) 2-arrow.

algebraic structure made up of 0, 1, 2 and 3-arrows with source, target and composition maps, subject to interchange identities that will not be given here. A lax 3-groupoid is a lax 3-category with all arrows strictly invertible. Taking as morphisms 3-functors preserving all the structure on the nose, they form the category $3 - Grpd$. See Berger's article for the construction of the adjoint pair to *SSet*, and the proof it creates a Quillen model structure.

It is unknown how this pattern(?) continues. That economical models of homotopy n-types will have something to do with nerves of weak n-categories may be only a low-dimensional illusion. At any rate, the known families of Quillen models for n-types, all $n \in \mathbb{N}$, seem to be based not on simplicial sets but simplicial groups. Note that the model structure on Simp(Gp) (whose homotopy theory is equivalent to that of reduced simplicial sets, thus that of connected spaces) is created by the forgetful functor to *SSet*; thus Thm. 2.4 does apply. We cull three examples from the expanding literature on the subject. For the first two (and the closely related n-hypergroupoids) see Cabello–Garzón [11] and Cabello [10] for the third.

Example 3.12. n-hypercrossed complexes: $n - \mathrm{HXC}(\mathrm{Gp}) \underset{J}{\overset{P}{\leftrightarrows}} \mathrm{Simp}(\mathrm{Gp})$

Example 3.13. n-fold simplicial groups: $\mathrm{Simp}^n(\mathrm{Gp}) \underset{\overline{\mathbb{W}}}{\overset{\mathbb{T}}{\leftrightarrows}} \mathrm{Simp}(\mathrm{Gp})$

Example 3.14. simplicial groups, with "truncated weak equivalences":

$$\mathrm{Simp}(\mathrm{Gp}) \underset{\mathrm{cosk}^n}{\overset{\mathrm{sk}^n}{\leftrightarrows}} \mathrm{Simp}(\mathrm{Gp})$$

Here are two examples where Thm. 2.4 applies coming from equivariant homotopy theory.

Example 3.15. G-equivariant spaces: $SSet^G \underset{R}{\overset{L}{\leftrightarrows}} SSet^I$

This example is taken from Dwyer–Kan [17] (which in fact deals with topological groups). Let G be a discrete group and $I := \{G_i\}$ a set of subgroups of G. The i^{th} component of the right adjoint R is the sub-*SSet* fixed by G_i; it creates what is sometimes called the "fine" equivariant homotopy theory of G-simplicial sets. Note that for Thm. 2.4 to apply, R must be *finite* limit definable, thus each G_i has to be finite. (There is no such restriction over *Set*.)

Remark 3.16. If $\mathcal{O}$ is the orbit category corresponding to the data G, $\{G_i\}$ — i.e. the full subcategory of G-sets with objects the cosets G/G_i — then the fine model structure on G-simplicial sets is Quillen-equivalent to simplicial presheaves on $\mathcal{O}$.

Example 3.17. cyclic sets: $\mathrm{Pre}(\Lambda) \leftrightarrows SSet^{\mathbb{N}}$

Let Λ be Connes' cyclic indexing category. A Quillen model structure has been established on $\mathrm{Pre}(\Lambda)$ by Spaliński [41], generalizing that of Dwyer–Hopkins–Kan [18]. For any positive integer r, there exists a combinatorial and in fact finite limit definable functor $\mathrm{Pre}(\Lambda) \xrightarrow{\Phi_r}$ *SSet* such that the geometric realization of $\Phi_r(X)$ is naturally homeomorphic to the $\mathbb{Z}/r$-fixed point set of the topological realization of X (which, recall, is an S^1-space). For any set of positive integers, the Φ_r collectively create a model structure on $\mathrm{Pre}(\Lambda)$ that is Quillen-equivalent to the corresponding fine homotopy theory of S^1-spaces, where weak equivalences and fibrations are detected on the $\mathbb{Z}/r\mathbb{Z}$-fixed subspaces.

Remark 3.18. No combinatorial model seems to be known for the finest version of S^1-equivariant homotopy theory, where a weak equivalence is a map that restricts to ordinary weak equivalences on the H-fixed subsets for every closed subgroup H of S^1 — that is to say, S^1 itself, in addition to the discrete ones.

For the next example, recall that Grothendieck defined abelian cohomology as the right derived functor of the global section functor, and it is an easy consequence of his foundational work on abelian categories that this exists for sheaves of modules over an arbitrary site. Quillen introduced his homotopy model formalism, in part, to allow for a calculus of non-abelian derived functors, e.g. from simplicial groups or rings. He asked in [39] whether this axiomatics is broad enough to apply to sheaves on an arbitrary site. After a 30-year hiatus, we see the answer is yes; moreover, it follows by an essentially tautologous extension of Quillen's original methods.

Example 3.19. simplicial $\mathbb{T}$-algebras: $SSet^{\mathbb{T}} \underset{U}{\overset{F}{\leftrightarrows}} SSet$

Here $\mathbb{T}$ is a finitary single-sorted equational universal algebraic theory, U the forgetful and F the free functor.

The case $\mathcal{E} = Set$ is due to Quillen [39]; to apply Cor. 2.4, one only needs the observation that a finitary equational algebraic theory is the same as a structure definable in terms of finite products (see e.g. Barr–Wells [2]) and of course the forgetful functor is definable, too.

Remark 3.20. For the case of simplicial rings and modules, this raises the possibility of a "purely homotopical" construction of the cotangent complex of a morphism of ringed topoi, even in the absence of enough points, when the problem was solved by Illusie [30].

Remark 3.21. Abelian groups, in particular, are a species of universal algebras, and Ex. 3.19 specializes to give a homotopy model structure on $\mathrm{Ab}(\mathcal{E})^{\Delta^{\mathrm{op}}}$, simplicial abelian sheaves, whose fibrations are the fibrations of the underlying simplicial sheaves. (A fibration of simplicial sheaves means here a "strong fibration", that is, a fibration in Joyal's model structure on simplicial sheaves.) Not every mono in $\mathrm{Ab}(\mathcal{E})^{\Delta^{\mathrm{op}}}$ is a cofibration. Via the Dold-Kan equivalence, this gives a model structure on $\mathrm{Ch}_{\mathbb{N}}(\mathrm{Ab}(\mathcal{E}))$, i.e. $\mathbb{N}$-indexed chain complexes, where the weak equivalences are the quasi-isomorphisms, but more maps are fibrations than just the injective ones. See Hovey [28] for related results.

As a last example of "creation by right adjoints", let $\mathcal{M}$ be a Quillen model category, $\mathcal{D}$ a diagram, and $\mathcal{D}_\delta$ the diagram $\mathcal{D}$ "made discrete", i.e. consisting of $\mathcal{D}$'s objects and identity arrows. The inclusion $\mathcal{D}_\delta \xrightarrow{i} \mathcal{D}$ induces an adjunction $\mathcal{M}^{\mathcal{D}} \underset{i^*}{\overset{L}{\leftrightarrows}} \mathcal{M}^{\mathcal{D}_\delta}$ that creates a model structure on $\mathcal{M}^{\mathcal{D}}$ under set-theoretic hypotheses on $\mathcal{M}$ (cofibrant generation; see Hirschhorn [27]). If $\mathcal{M}$ was a coherently definable homotopy theory, this endows $\mathcal{M}^{\mathcal{D}}$ with a sheafifiable model structure. (One can apply Prop. 2.3 directly.)

Remark 3.22. Let $\mathcal{M}$ be a Quillen model category, $\mathcal{D}$ a diagram. Let us agree that weak equivalences in $\mathcal{M}^{\mathcal{D}}$ are to be the natural transformations that are $\mathcal{D}$-objectwise weak equivalences. The existence of a Quillen model structure on $\mathcal{M}^{\mathcal{D}}$ extending this seems to be a rather muddy affair. It is known to hold, unconditionally in $\mathcal{M}$, for combinatorially distinguished $\mathcal{D}$: for example, those satisfying the Reedy property — see e.g. Hovey [29] — or having a simplicially finite nerve — see Dwyer–Spaliński [14], Franke [19]. As pointed out above, it also holds, unconditionally in $\mathcal{D}$, for set-theoretically distinguished $\mathcal{M}$. If both $\mathcal{D}$ and $\mathcal{M}$ are distinguished, the two constructions need not coincide. For yet more special — for example, sheafifiable — $\mathcal{M}$ and arbitrary $\mathcal{D}$, $\mathcal{M}^{\mathcal{D}}$ will possess *two* distinct model structures, in analogy with Heller's [25] "left" and "right"model structures for simplicial diagrams. The *cosimplicial spaces* of Bousfield–Kan [9], i.e. cosimplicial simplicial sets, possess (at least) three distinct cofibration classes for the same choice of weak equivalences, and they all survive to diagrams of cosimplicial spaces. [3] attempts to put some order in this zoo; it is proven that for a wide class of model categories (see therein for the precise condition) all possible small-generated cofibrations classes yield Quillen-equivalent homotopy theories. Note that the theory of homotopy limits and colimits — which is the chief reason to study $\mathcal{M}^{\mathcal{D}}$ — can be developed bypassing the question of existence of a full model structure on $\mathcal{M}^{\mathcal{D}}$; see Dwyer–Hirschhorn–Kan [15] and Chachólski–Scherer [12] for two approaches.

There are variants on Thm. 2.4 that will not be packed into a meta-theorem here. For example, it is not necessary that the class of cofibrations $\mathsf{C}(\mathcal{E})$ to be lifted is coherently definable; it suffices if it is of the form $\mathsf{cof}(I)$, with I a coherently definable class. This allows one to sheafify Hinich's [26] model structures on dg operads, algebras and modules. Also, given one cofibrantly generated model structure on a locally presentable category, Jeff Smith's theorem (quoted as 4.1 below) allows for an easy argument to pass to a *smaller* (though still set-generated) class of cofibrations. In essence, it suffices if the proposed class of cofibrations works in *Set*, and is dominated by one to which 2.4 applies. This helps in comparing work of Joyal–Tierney [33] on simplicial groupoids with that of Crans [13], both written already in the setting of sheaves.

4. Localization along a Quillen left adjoint

Let us sum up the main properties of the Quillen model categories encountered in the two parts of this paper.

- The underlying category of models is $\mathsf{S}(\mathcal{E})$, the category of S-structures in a topos $\mathcal{E}$, where S is finite limit definable.
- The subcategory $\mathsf{W}_{\mathcal{E}}$ of weak equivalences can be specified by a set of coherent axioms in the language of morphisms of S-structures.
- The class of cofibrations $\mathsf{C}_{\mathcal{E}}$ is functorial in $\mathcal{E}$, and preserved by inverse images of geometric morphisms. It is small-generated, i.e. $\mathsf{C}_{\mathcal{E}} = \mathsf{cof}(I_{\mathcal{E}})$ for a set $I_{\mathcal{E}}$ depending (non-canonically) on $\mathcal{E}$.
- For any Grothendieck topos $\mathcal{E}$, the data provide a cofibrantly generated Quillen model category. (Fixing only S and W, there may exist several suitable cofibration classes.)

Goerss and Jardine [21], working with E_*-local simplicial objects, where E_* is a homology theory, proved that given a geometric morphism $\mathcal{E} \xrightarrow{f} \mathcal{F}$, one can take as weak equivalences in $\mathcal{F}^{\Delta^{\mathrm{op}}}$ the maps that f^* takes into weak equivalences in $\mathcal{E}^{\Delta^{\mathrm{op}}}$. (Cofibrations in $\mathcal{F}^{\Delta^{\mathrm{op}}}$ stay the same, i.e. are all monomorphisms.) The Quillen model structure thus obtained on $\mathcal{F}^{\Delta^{\mathrm{op}}}$ extends the class of weak equivalences, so is a "localization" of the original. We will see that this phenomenon extends to any sheafifiable homotopy theory. Earlier, Jardine [31] observed that when $\mathcal{E}^{\Delta^{\mathrm{op}}} \xrightarrow{f} \mathcal{F}^{\Delta^{\mathrm{op}}}$ is the inclusion of simplicial sheaves on a site into simplicial presheaves, the model structure induced on simplicial presheaves via sheafification is Quillen equivalent to the (canonical) one on simplicial sheaves. This holds for any sheafifiable homotopy theory as well. Neither fact is specific to sheaves; they follow from a very robust statement about Quillen model categories whose underlying category is locally presentable. Recall the following version of J. Smith's theorem (cf. Thm. 1.7, Prop. 1.15 and Prop. 1.19 of part I):

Theorem 4.1. *Let $\mathcal{C}$ be a locally presentable category,* W *a full accessible subcategory of* $\mathrm{Mor}(\mathcal{C})$*, and I a set of morphisms of $\mathcal{C}$. Suppose they satisfy:*

- **(0)** W *has the 2-of-3 property (Quillen's axiom* ***M2****).*
- **(1)** $\mathsf{inj}(I) \subseteq \mathsf{W}$.
- **(2)** *The class* $\mathsf{cof}(I) \cap \mathsf{W}$ *is closed under transfinite composition and under pushout.*

Then setting weak equivalences:=W*, cofibrations:=*$\mathsf{cof}(I)$ *and fibrations:=*$\mathsf{inj}(\mathsf{cof}(I) \cap \mathsf{W})$*, one obtains a cofibrantly generated Quillen model structure on $\mathcal{C}$.*

Proposition 4.2. *Let $\mathcal{C}_1$, W_1, I_1 and $\mathcal{C}_2$, W_2, I_2 be data satisfying the hypotheses of Thm. 4.1, and $\mathcal{C}_1 \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{C}_2$ a Quillen adjunction. Assume in addition that L takes weak equivalences in $\mathcal{C}_2$ into weak equivalences in $\mathcal{C}_1$. There exists a Quillen model structure on $\mathcal{C}_2$ with cofibrations* $\mathsf{cof}(I_2)$ *and weak equivalences* $\mathsf{W}_L := \{g \in \mathcal{C}_2 \mid L(g) \in \mathsf{W}_1\}$.

Proof. W_L is an accessible class of maps, since it is the inverse image of an accessible category by an accessible functor. $\mathsf{W}_2 \subseteq \mathsf{W}_L$ by assumption. L preserves cofibrations and colimits, and with these the criteria of Thm. 4.1 are verified. □

Corollary 4.3. *Let* $\mathsf{S}(-)$, $\mathsf{W}_{(-)}$, $\mathsf{C}_{(-)}$ *be a sheafifiable homotopy model theory, $\mathcal{E} \xrightarrow{f} \mathcal{F}$ a geometric morphism. There exists a Quillen model structure on* $\mathsf{S}(\mathcal{F})$ *with cofibrations* $\mathsf{C}_{\mathcal{F}}$ *and weak equivalences* $\mathsf{W}_{f^*} := \{g \in \mathsf{S}(\mathcal{F}) \mid f^*(g) \in \mathsf{W}_{\mathcal{E}}\}$.

Proposition 4.4. *Suppose that, in the situation of 4.2, $\mathcal{C}_1$ is actually a full reflexive subcategory of $\mathcal{C}_2$ via $L \dashv R$. Then this adjunction provides a Quillen equivalence between $\mathcal{C}_1$ (with cofibrations* $\mathsf{cof}(I_1)$ *and weak equivalences* W_1*) and $\mathcal{C}_2$ with cofibrations* $\mathsf{cof}(I_2)$ *and weak equivalences* W_L.

Proof. L preserves cofibrations by assumption, and takes W_L into W_1 by definition; so $L \dashv R$ is a Quillen pair. To prove it a Quillen equivalence, it is enough to show that the reflector natural transformations $X \to RL(X)$ are weak equivalences in $\mathcal{C}_2$. But they are sent into isomorphisms by L, so that certainly holds. □

Corollary 4.5. *With $\mathsf{S}(-)$, $\mathsf{W}_{(-)}$, $\mathsf{C}_{(-)}$ as in 4.4, let j be a Lawvere–Tierney topology on a topos $\mathcal{F}$ and let f be the canonical topos inclusion $\mathrm{Sh}_j(\mathcal{F}) \hookrightarrow \mathcal{F}$. $f^* \dashv f_*$ induces a Quillen equivalence between $\mathsf{S}(\mathrm{Sh}_j(\mathcal{F}))$ (with cofibrations $\mathsf{C}_{\mathrm{Sh}_j(\mathcal{F})}$ and weak equivalences $\mathsf{W}_{\mathrm{Sh}_j(\mathcal{F})}$) and $\mathsf{S}(\mathcal{F})$ with cofibrations $\mathsf{C}_{\mathcal{F}}$ and weak equivalences W_{f^*}.*

References

[1] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories.* Number 189 in London Math. Soc. Lecture Note Series. Cambridge University Press, 1994.

[2] Michael Barr and Charles Wells. *Toposes, triples and theories.* Number 278 in Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Science). Springer-Verlag, 1985.

[3] Tibor Beke. How (non-)unique is the choice of cofibrations? See http://www.math.lsa.umich.edu/˜tbeke.

[4] Tibor Beke. *Homotopy theory and topoi.* PhD thesis, Massachusetts Institute of Technology, 1998.

[5] Tibor Beke. Sheafifiable homotopy model categories. *Mathematical Proceedings of the Cambridge Philosophical Society*, 129(3):447–475, 2000.

[6] Clemens Berger. Double loop spaces, braided monoidal categories and algebraic 3-type of space. In *Higher homotopy structures in topology and mathematical physics*, number 227 in Contemporary Mathematics, pages 49–66. American Mathematical Society, 1999.

[7] David Blanc. New model categories from old. *Journal of Pure and Applied Algebra*, 109(1):37–60, 1996.

[8] Francis Borceux. *Handbook of Categorical Algebra, vol.I-III.* Cambridge University Press, 1994.

[9] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*, volume 304 of *Lecture Notes in Mathematics*. Springer-Verlag, 1972.

[10] Julia G. Cabello. Simplicial groups as models for n-types. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 38(1):67–92, 1997.

[11] Julia G. Cabello and Antonio R. Garzón. Closed model structures for algebraic models of n-types. *Journal of Pure and Applied Algebra*, 103(3):287–302, 1995.

[12] Wojciech Chachólski and Jérôme Scherer. Homotopy meaningful constructions: Homotopy colimits. Max-Planck-Institut für Mathematik, preprint no. 44, 1998.

[13] Sjoerd E. Crans. Quillen closed model structures for sheaves. *Journal of Pure and Applied Algebra*, 101(1):35–57, 1995.

[14] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, 1995.

[15] William G. Dwyer, Philip S. Hirschhorn, and Daniel M. Kan. Model categories and more general abstract homotopy theory. See http://www-math.mit.edu/˜psh.

[16] William G. Dwyer and Daniel M. Kan. Homotopy theory and simplicial groupoids. *Indagationes Mathematicae (Nederl. Akad. Wetensch. Proc. Ser. A)*, 46(4):379–385, 1984.

[17] William G. Dwyer and Daniel M. Kan. Singular functors and realization functors. *Indagationes Mathematicae (Nederl. Akad. Wetensch. Proc. Ser. A)*, 46(2):147–153, 1984.

[18] Willian G. Dwyer, Michael J. Hopkins, and Daniel M. Kan. The homotopy theory of cyclic sets. *Transactions of the American Mathematical Society*, 291(1):281–289, 1985.

[19] Jens Franke. Uniqueness theorems for certain triangulated categories with an Adams spectral sequence. See http://www.math.uiuc.edu/K-theory/.

[20] Henri Gillet and Christophe Soulé. Filtrations on higher algebraic K-theory. In *Algebraic K-theory (Seattle, 1997)*, number 67 in Proceedings of Symposia in Pure Mathematics, pages 89–148. American Mathematical Society, 1999.

[21] Paul Goerss and J. F. Jardine. Localization theories for simplicial presheaves. *Canadian Journal of Mathematics*, 50(5):1048–1089, 1998.

[22] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory.* Number 174 in Progress in Mathematics. Birkhäuser Verlag, 1999.

[23] Marek Golasiński. The category of cubical sets and the category of small categories. *Bull. Acad. Polon. Sci. Sr. Sci. Math.*, 27(11):941–945, 1981.

[24] Marek Golasiński. On weak homotopy equivalence of simplicial and cubical nerve. *Demonstratio Math.*, 14(4):889–899, 1982.

[25] Alex Heller. *Homotopy theories.* Number 383 in Memoirs of the AMS. American Mathematical Society, 1988.

[26] Vladimir Hinich. Homological algebra of homotopy algebras. arXiv:q-alg/9702015.

[27] Philip Hirschhorn. Localization of model categories. See http://www-math.mit.edu/~psh.

[28] Mark Hovey. Model category structures on chain complexes of sheaves. See http://hopf.math.purdue.edu/.

[29] Mark Hovey. *Model categories.* Number 63 in Mathematical Surveys and Monographs. American Mathematical Society, 1999.

[30] Luc Illusie. *Complexe cotangent et déformations, vol.I-II.* Number 239,283 in Lecture Notes in Mathematics. Springer-Verlag, 1971.

[31] J. F. Jardine. Simplicial presheaves. *Journal of Pure and Applied Algebra*, 47(1):35–87, 1987.

[32] André Joyal and Myles Tierney. Strong stacks and classifying spaces. In *Category theory (Como, 1990)*, number 1488 in Lecture Notes in Mathematics, pages 213–236. Springer-Verlag, 1991.

[33] André Joyal and Myles Tierney. On the homotopy theory of sheaves of simplicial groupoids. *Mathematical Proceedings of the Cambridge Philosophical Society*, 120(2):263–290, 1996.

[34] André Joyal and Gavin Wraith. Eilenberg-MacLane toposes and cohomology. In *Mathematical applications of category theory*, number 30 in Contemporary Mathematics, pages 117–131. American Mathematical Society, 1984.

[35] Michael Makkai. Generalized sketches as a framework for completeness theorems, part I–III. *Journal of Pure and Applied Algebra*, 115(1,2,3):49–274, 1997.

[36] Michael Makkai and Gonzalo E. Reyes. *First order categorical logic. Model-theoretical methods in the theory of topoi and related categories.* Number 611 in Lecture Notes in Mathematics. Springer-Verlag, 1977.

[37] Ieke Moerdijk. Bisimplicial sets and the group-completion theorem. In *Algebraic K-theory: connections with geometry and topology*, number 279 in NATO Advanced Sci. Inst. Series, pages 225–240. Kluwer, 1989.

[38] Ieke Moerdijk and Jan-Alve Svensson. Algebraic classification of equivariant homotopy 2-types, I. *Journal of Pure and Applied Algebra*, 89(1):187–216, 1993.

[39] Daniel Quillen. *Homotopical algebra.* Number 43 in Lecture Notes in Mathematics. Springer-Verlag, 1967.

[40] Charles Rezk. *Spaces of algebra structures and cohomology of operads.* PhD thesis, Massachusetts Institute of Technology, 1996.

[41] Jan Spaliński. Strong homotopy theory of cyclic sets. *Journal of Pure and Applied Algebra*, 99(1):35–52, 1995.

[42] Ross H. Street. The algebra of oriented simplexes. *Journal of Pure and Applied Algebra*, 49(3):283–335, 1987.

[43] R. W. Thomason. Cat as a closed model category. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 21(3):305–324, 1980.

Mathematical Institute, Utrecht University, Utrecht, Netherlands

Current address: Department of Mathematics, University of Michigan, East Hall, 525 East University Avenue, Ann Arbor, MI 48109-1109, U.S.A.

E-mail address: `tbeke@umich.edu`