16.513 Control Systems

Controllability and Observability
(Chapter 6)

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A General Framework in State-Space Approach

Given an LTI system:

$$\dot{x} = Ax + Bu; \quad y = Cx \qquad (*)$$

The system might be unstable or doesn't meet the required performance spec. How can we improve the situation?

The main approach: Let u= v- Kx (state feedback), then

$$\dot{x} = Ax + B(v - Kx); \quad y = Cx + D(v - Kx)$$
$$= (A - BK)x + Bv; \quad = (C - DK)x - Dv$$

The performance of the system is changed by matrix K.

Questions:

- Is there a matrix K s.t. A-BK is stable?
- Can eig(A-BK) be moved to desired locations?

These issues are related to the controllability of (*)

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Main Result 1: The eigenvalues of A-BK can be moved to any desired locations iff the system (*) is controllable.

Another situation: the state x is not completely available. Only a linear combination of x, e.g., $y = Cx$, can be measured. How can we realize $u = v - Kx$?

A possible solution: build an observer to estimate x based on measurement of y.

Main result 2: The observer error (difference between the real x and estimated x) can be made arbitrarily small within arbitrarily short time period iff (*) is observable.

We will arrive at these conclusions in Chapter 8. Before that, we need to prepare some tools and go through these fundamental problems: controllability and observability. 3

Controllability: Definition

Consider the system

$$\dot{x} = Ax + Bu, \quad x \in R^n; \; u \in R^p.$$

Controllability is a relationship between state and input.

Definition: The system, or the pair (A,B), is said to be controllable if for any initial state $x(0)=x_0$ and any final state x_d, there exist a finite time $T > 0$ and an input $u(t)$, $t \in [0,T]$ such that

$$x(T) = e^{AT}x_0 + \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau = x_d \qquad (1)$$

Comment: There may exist different T and u that satisfy (1). As a result, there may be different trajectories starting from x_0 and end at x_d. Controllability does not care about the difference. 4

$$W_c(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau = \int_0^t e^{A(t-\tau)} BB' e^{A'(t-\tau)} d\tau$$

Equivalent conditions: The following are equivalent conditions for the pair (A,B) to be controllable:

1) $W_c(t)$ is nonsingular for every $t > 0$.
2) $W_c(t)$ is nonsingular for at least one $t > 0$.
3) For every $v \in R^n$, $v \neq 0$, $v' e^{At} B$ is not identically zero.
4) The matrix $G^c = [B \;\; AB \;\; A^2B \; \ldots \; A^{n-1}B]$ has full row rank, i.e., $\rho(G^c) = n$.
5) The matrix $M(\lambda) = [A - \lambda I \;\; B]$ has full row rank at all $\lambda \in C$.
6) $M(\lambda)$ has full row rank at every eigenvalues of A.

Note: $M(\lambda)$ has full row rank if λ is not an eigenvalue of A. We only need to check the rank of $M(\lambda)$ at eigenvalues of A.

Note : Of all the conditions, only 4) and 6) can be practically verified.

Example: Determine the controllability for

$$\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} a \\ b \end{bmatrix}$$

Approach 1: $G^c = [B \;\; AB] = \begin{bmatrix} a & -a \\ b & -b \end{bmatrix}$

$\rho(G) < 2 = n$ for all possible a and b

The system not controllable whatever a and b are.

Approach 2: Check $M(\lambda) = [A - \lambda I \;\; B]$ at $\lambda = -1$

$$M(-1) = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \end{bmatrix}$$

$\rho(M(-1)) < 2$ for all possible a and b

Same conclusion on controllability

Example:

$$\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} a \\ b \end{bmatrix}$$

Approach 1: $G^c = [B \;\; AB] = \begin{bmatrix} a & -a \\ b & -2b \end{bmatrix}$

$$\det G^c = -ab, \quad \begin{cases} \rho(G^c) = 2, & \text{if } a \neq 0 \text{ and } b \neq 0 \\ \rho(G^c) < 2, & \text{if either } a = 0 \text{ or } b = 0 \end{cases}$$

The system is controllable if $a \neq 0$ and $b \neq 0$.

Approach 2: Check M(λ)=[A-λI B] at λ=-1

$$M(-1) = \begin{bmatrix} 0 & 0 & a \\ 0 & -1 & b \end{bmatrix}, \quad M(-2) = \begin{bmatrix} 1 & 0 & a \\ 0 & 0 & b \end{bmatrix},$$

$\rho(M(-1)) = \rho(M(-2)) = 2$ iff $a \neq 0$ and $b \neq 0$

Same conclusion on controllability

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A general SI system (diagonalizable)

$$\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

The above system is controllable if and only if
the eigenvalues are distinct and none of the b_i's is zero

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Example:

$$\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$G^c = [B \;\; AB] = \begin{bmatrix} b_1 & \alpha b_1 - \beta b_2 \\ b_2 & \beta b_1 + \alpha b_2 \end{bmatrix}$$

$$\det G^c = \beta(b_1^2 + b_2^2), \qquad \begin{cases} \rho(G^c) = 2, & \text{if } \beta \neq 0 \text{ and } b_1^2 + b_2^2 \neq 0 \\ \rho(G^c) < 2, & \text{if either } \beta = 0 \text{ or } b_1^2 + b_2^2 = 0 \end{cases}$$

The system is controllable if $\beta \neq 0$ and $(b_1, b_2) \neq (0,0)$

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Theorem: Consider the pair

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix}$$

Suppose that the eigenvalues of A_i and those of A_j are disjoint for $i \neq j$. Then (A,B) is controllable iff (A_i,B_i) is controllable for all i.

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Theorem: Let $\rho(B) = p$. The pair (A,B) is controllable iff

$$G^c_{n-p+1} := [B \;\; AB \;\; A^2B \;\dots\; A^{n-p}B]$$

has full row rank. This is equivalent to $G^c_{n-p+1}G^{c\prime}_{n-p+1}$ being nonsingular, and to $G^c_{n-p+1}G^{c\prime}_{n-p+1} > 0$ (positive definite.)

Example:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$n=4$, $p=2$. $\rho(B)=2=p$.

$$G^c_{n-p+1} = [B \;\; AB \;\; A^2B] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 & 0 & -4 \end{bmatrix}$$

$$\begin{matrix} b_1 & b_2 & Ab_1 & Ab_2 & A^2b_1 & A^2b_2 \end{matrix}$$

The first 4 columns are LI. $\Rightarrow \rho(G^c_{n-p+1})=4=n$

$\Rightarrow$ (A,B) controllable

Example:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{n=4, p=2. } \rho(B)=2.$$

$$G^c_{n-p+1} = \left[B \ \ AB \ \ A^2B\right] = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 8 & 0 \\ 1 & 0 & 2 & 0 & 4 & 0 \\ 1 & 1 & 3 & 3 & 9 & 9 \end{bmatrix}$$

$$\qquad\qquad\qquad\qquad b_1 \quad b_2 \quad Ab_1 \quad Ab_2 \quad A^2b_1 \quad A^2b_2$$

The first 3 columns are LI.
The 4^{th} is dependent on the first 3.

$$\left[b_1 \ b_2 \ Ab_1 \ A^2b_1\right] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 3 & 8 \\ 1 & 0 & 2 & 4 \\ 1 & 1 & 3 & 9 \end{bmatrix} \text{ has full row rank}$$

Hence (A,B) is controllable.

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Effect of equivalence transformation

Recall that equivalence transformation can make the structure cleaner and simplify analysis.

Question:
Does similarity transformation retain the controllability property?

Theorem: The controllability property is invariant under any equivalence transformation

Proof: Consider (A,B) with $G^c=[B\ AB\ A^2B\ \ldots.\ A^{n-1}B]$.
Let the transformation matrix be P. Then $(A,B) \Leftrightarrow (PAP^{-1}, PB)$

$$\begin{aligned} \overline{G}^c &= [\overline{B} \ \ \overline{A}\overline{B} \cdots \overline{A}^{n-1}\overline{B}] \\ &= [PB \ \ PAP^{-1}PB \ \cdots \ PA^{n-1}P^{-1}PB] \\ &= [PB \ \ PAB \ \cdots \ PA^{n-1}B] \\ &= P[B \ \ AB \ \cdots \ A^{n-1}B] \\ &= PG^c \end{aligned}$$

Since P is nonsingular,

$$\rho(\overline{G}^c) = \rho(G^c)$$

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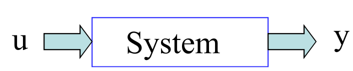
Next Problem: Observability

Observability: A dual concept

Consider an n-dimensional, p-input, q-output system:

$$\dot{x} = Ax + Bu; \quad y = Cx + Du$$



Assume that we know the input and can measure the output, but has no access to the state.

Definition: The system, is said to be observable if for any unknown initial state x(0), there exists a finite $t_1 > 0$ such that x(0) can be exactly evaluated over $[0, t_1]$ from the input u and the output y. Otherwise the system is said to be unobservable.

Duality between controllability and observability

Theorem of duality: The pair (A,B) is controllable if and only if $(A_1, C_1) = (A', B')$ is observable.

$$\dot{x} = Ax + Bu \quad \xleftrightarrow{\text{Dual systems}} \quad \begin{aligned} \dot{z} &= A_1 z = A' z \\ y &= C_1 z = B' z \end{aligned}$$

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Equivalent conditions for observability:

1) The pair (A,C) is observable.
2) $W_o(t)$ is nonsingular for some $t > 0$.
3) The observability matrix

$$G^o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full column rank, i.e., $\rho(G^o) = n$.

4) The matrix

$$M^o(\lambda) = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$$

has full column rank at every eigenvalue of A.

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Theorem: The pair (A,C) is observable if and only if

$$G^{o}_{n-q+1} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-q} \end{bmatrix}$$

has full column rank, where q=ρ(C).

Theorem: The obserbability property is invariant under any equivalence transformation;

Theorem: Consider the pair

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & A_m \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 & \cdots & C_m \end{bmatrix}$$

Suppose that the eigenvalues of A_i and those of A_j are disjoint for .$i \neq j$. Then (A,C) is observable iff (A_i, C_i) is observable for all i.

So far, we have learned

- Controllability
- Observability

Next, we will study

- Canonical decomposition: to divide the state space into controllable/uncontrollable, observable/unobservable subspaces

Canonical Decomposition

Consider an LTI system,

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

Let z = Px, where P is nonsingular, then

$$\dot{z} = \overline{A}x + \overline{B}u, \quad y = \overline{C}z + \overline{D}u$$

where $\overline{A} = PAP^{-1}, \ \overline{B} = PB, \quad \overline{C} = CP^{-1}, \quad \overline{D} = D$

Recall that under an equivalence transformation, all properties, such as stability, controllability and observability are preserved.

We also have $\overline{G}^c = PG^c, \quad \overline{G}^o = G^o P^{-1}$

Next we are going to use equivalence transformation to obtain certain specific structures which reflect controllability and observability.

Controllability decomposition

Recall $G^c=[B\ AB\ \dots\ A^{n-1}B]$. Suppose that $\rho(G^c) = n_1 < n$.
Then G^c has at most n_1 LI columns.
They form a basis for the range space of G^c.

Theorem: Suppose that $\rho(G^c) = n_1 < n$. Let Q be a nonsingular matrix whose first n_1 columns are LI columns of G^c. Let $P=Q^{-1}$. Then

$$\overline{A} = PAP^{-1} = \begin{bmatrix} \overline{A}_c & \overline{A}_{12} \\ 0 & \overline{A}_{\bar{c}} \end{bmatrix}, \quad \overline{B} = PB = \begin{bmatrix} \overline{B}_c \\ 0 \end{bmatrix}, \quad \overline{A}_c \in R^{n_1 \times n_1}, \overline{B}_c \in R^{n_1 \times p}$$

$$\overline{C} = \begin{bmatrix} \overline{C}_c & \overline{C}_{\bar{c}} \end{bmatrix}$$

Moreover, the pair $(\overline{A}_c, \overline{B}_c)$ is controllable and

$$\overline{C}_c (sI - \overline{A}_c)^{-1} \overline{B}_c + D = C(sI - A)^{-1} B + D$$

See page 159 for the proof.

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Discussion:

After state transformation, the equivalent system is

$$\begin{aligned} \dot{z}_1 &= \overline{A}_c z_1 + \overline{A}_{12} z_2 + \overline{B}_c u \\ \dot{z}_2 &= \overline{A}_{\bar{c}} z_2 \end{aligned}$$

The input u has no effect on z_2. This part of state is uncontrollable. The first sub-system is controllable if $z_2=0$. If $z_2 \neq 0$, then

$$z_1(t_1) = e^{\overline{A}_c t_1} z_{10} + \int_0^{t_1} e^{\overline{A}_c (t_1 - \tau)} \overline{B}_c u(\tau) d\tau + \int_0^{t_1} e^{\overline{A}_c (t_1 - \tau)} \overline{A}_{12} z_2(\tau) d\tau$$

$$z_2(\tau) = e^{\overline{A}_{\bar{c}} \tau} z_{20}$$

Given a desired value for z_1, say z_{1d}. If we let

$$v(t_1) = \int_0^{t_1} e^{\overline{A}_c (t_1 - \tau)} \overline{A}_{12} e^{\overline{A}_{\bar{c}} \tau} z_{20} d\tau, \quad \overline{W}_c(t_1) = \int_0^{t_1} e^{\overline{A}_c \tau} \overline{B}_c \overline{B}_c' e^{\overline{A}_c' \tau} d\tau$$

and $u(t) = -\overline{B}_c' e^{\overline{A}_c'(t_1 - t)} \overline{W}_c^{-1}(t_1)[e^{\overline{A}_c t_1} z_{10} + v(t_1) - z_{1d}]$

Then you can verify that $z_1(t_1)=z_{1d}$.

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Example: $\dot{x} = Ax + Bu$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

n=3, p=2, n-p+1=2.
Only need to check G^c_2

$$G_2^c = [B \quad AB] = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad \rho(G_2^c) = 2 < 3, \quad \text{uncontrollable}$$

$$\text{Let } Q = [b_1\ b_2\ q] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ P = Q^{-1}$$

q is picked to make Q nonsingular

$$\bar{A} = PAQ = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix},$$

$$\bar{B} = PB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}$$

Note: the last column of Q is different from the book (page 161).
As a result, $\bar{A}_{12}$ is different from that in the book, which is 0.

Observability decomposition (follows from duality)

$$\text{Recall} \quad G^o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Theorem: Suppose that $\rho(G^o) = n_1 < n$. Let P be a nonsingular matrix whose first n_1 rows are LI rows of G^o. Then

$$\bar{A} = PAP^{-1} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix}, \quad \bar{B} = PB = \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix}, \quad \bar{A}_o \in R^{n_1 \times n_1}, \bar{B}_o \in R^{n_1 \times p}$$

$$\bar{C} = [\bar{C}_o \quad 0], \qquad \bar{C}_o \in R^{q \times n_1}$$

Moreover, the pair $(\bar{A}_o, \bar{C}_o)$ is observable and
$\bar{C}_o(sI - \bar{A}_o)^{-1}\bar{B}_o + D = C(sI - A)^{-1}B + D$

Discussion: After state transformation, the equivalent system is

$$\begin{aligned} \dot{z}_1 &= \bar{A}_o z_1 + \bar{B}_o u \\ \dot{z}_2 &= \bar{A}_{21} z_1 + \bar{A}_{\bar{o}} z_2 + \bar{B}_{\bar{o}} u, \\ y &= \bar{C}_o z_1 + Du \end{aligned}$$

z_2 may be affected by z_1
but has no effect on y or z_1

Summary for today:

- Controllability
- Observability
- Canonical decomposition
  - Controllable/uncontrollable
  - Observable/unobservable

Next Time:

- Controllability and observability continued
  - Controllability/observability decomposition
  - Minimal realization
  - Conditions for Jordan form conditions
  - Parallel results for discrete-time systems
  - Controllability after sampling
- State feedback design (introduction)

Problem Set #9

1. Is the following state equation controllable? observable?

$$\dot{x} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x$$

If not controllable, reduce it to a controllable one;
If not observable, reduce it to an observable one.

2. Is the following state equation controllable? observable?

$$\dot{x} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} x$$

If not controllable, reduce it to a controllable one;
If not observable, reduce it to an observable one.