Free Vibration of Single-Degree-of-Freedom (SDOF) Systems

- ## Procedure in solving structural dynamics problems

  1. **Abstraction/modeling** – Idealize the actual structure to a simplified version, depending on the purpose of analysis.
  2. **Derivation** – Derive the dynamic governing equation of the simplified system.
  3. **Evaluation** – Solve the dynamic governing equation for the DOF of interest and for dynamic parameters.
  4. **Analysis** – Interpret the result and/or compare with design values.

- ## Approaches for deriving dynamic governing equations

  – **Dynamic equilibrium** – Based on Newton's second law of motion and D'Alembert's principle, dynamic equilibrium is achieved by balancing the external loading with resistant forces including a fictitious inertia force, a damping force, and an elastic force acting on a moving free body (or DOF).

    * D'Alembert's principle – Sum of all forces acting on a moving free body must equal to a fictitious inertia force.

      $$\Sigma(f_x) - m\ddot{u} = 0 \tag{1}$$

      where f_x represents external loading, damping force, and elastic force.

    * Example of a SDOF mass-damper-spring system

    * (Q: What is the limit in this approach?)

– **Lagrange's equations of motion** – In a dynamical system, the change of the Lagrangian for each generalized co-ordinate equals to the time derivative of the velocity change of the Lagrangian.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \tag{2}$$

where $L = T - U$ is the Lagrangian of the system, $T = \frac{1}{2}m\dot{q}_i{}^2$ is the kinetic energy, U is the potential energy, q_i is the generalized co-ordinate at i ($1 \leq i \leq n$, n is the number of DOF), $\dot{q}_i = \frac{dq_i}{dt}$ is the velocity of the generalized co-ordinate, and Q_i is the generalized force at i, which excludes the forces considered in U. In nonconservative dynamical systems, Eq.(2) becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} = Q_i \tag{3}$$

where F is the Rayleigh's dissipative function.

* Example of a SDOF mass-damper-spring system

* (Q: What is the limit in this approach?)

– **Hamilton's principle** – In Hamilton's own words,

> *Of all the possible paths along which a dynamical system may move from one point to another within a specified time integral (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.*
> — W.R. Hamilton (1834)

Mathematically, for conservative dynamical systems, Hamilton's principle is read as follows.

$$\int_{t_1}^{t_2} \delta\left[T(t) - U(t)\right] dt = 0 \tag{4}$$

where δ is the virtual displacement as defined in the Principle of Virtual Work. Since $T = T(t) = T(t, \dot{q}_i)$ and $U = u(t) = U(t, q_i)$, we have

$$\delta \int_{t_1}^{t_2} L\left(q_i, \dot{q}_i\right) dt = 0 \tag{5}$$

Therefore, it is also equivalent to state that:

> Of all the possible paths along which a dynamical system may move from one point to another in configuration space within a specified time integral, the actual path followed is that which minimizes the time integral of the Lagrangian for the system.

Similarly, for non-conservative dynamic systems, Eq.(4) becomes

$$\int_{t_1}^{t_2} \delta\left[T(t) - U(t)\right] dt + \int_{t_1}^{t_2} \delta W_{nc}(t) dt = 0 \tag{6}$$

Hamilton's principle is also written in the following form.

$$\delta W = 0 \tag{7}$$

which states that, in a stable system, the total work/energy done by all considered forces in the system, under the perturbation of a virtual displacement, must be zero.

∗ Example of a SDOF mass-damper-spring system

∗ (Q: What is the limit in this approach?)

- **Evaluation of dynamic governing equations**
  - Many physical laws and relations appear mathematically in the form of differential equations (DEs).
  - The governing equation of dynamical systems is usually either an ordinary differential equation (ODE) or a partial differential equation (PDE).

  - Evaluating dynamic governing equations is equivalent to solving DEs.

Further information on the evaluation of ODE and PDE can be found in Agnew (1942), Spiegel (1958), Boyce and DiPrima (1969), and Kreyszig (2006). (Review your knowledge from *Differential Equations*.)

- **Undamped SDOF systems**
  - Undamped: There is no damping considered in the systems.
  - SDOF: The number of generalized coordinates in the systems is unity.
  - Free vibration is always generated by:
    1. initial displacement, $u(t=0)=u_0$, or
    2. initial velocity, $\dot{u}(t=0)=\dot{u}_0$, or
    3. combination of the above two.
  - Solution to the free vibration problem (ODE) of undamped SDOF systems is a particular solution to the ODE.
  - Governing equation of an undamped SDOF mass-spring system:

$$m\ddot{u}+ku=0 \tag{8}$$

Solution to Eq.(8) is

$$u=u(t)=A\sin(\omega_n t)+B\cos(\omega_n t) \tag{9}$$

where

$$\omega_n=\sqrt{\frac{k}{m}} \tag{10}$$

is the natural frequency (rad/s) of the undamped SDOF system. A and B are integration constants determined by initial conditions of the problem. Let $u(0)=u_0$ to be the initial displacement and $\dot{u}(0)=\dot{u}_0$ the initial velocity. We then can replace constants A and B by

$$B=u_0 \tag{11}$$

$$A=\frac{\dot{u}_0}{\omega_n} \tag{12}$$

Therefore, the complete solution of an undamped SDOF system in free vibration is

$$\boxed{u = u(t) = \frac{\dot{u}_0}{\omega_n}\sin(\omega_n t) + u_0\cos(\omega_n t)} \tag{13}$$

The amplitude of this dynamic displacement function is

$$|u| = \sqrt{\left(\frac{\dot{u}_0}{\omega_n}\right)^2 + (u_0)^2} \tag{14}$$

(**Note the difference between the notation of vibration amplitude defined here and the one used in the textbook.**)

- Natural frequency: The natural frequency of this undamped SDOF system is

$$\omega_n = \sqrt{\frac{k}{m}}$$

- Complex plane representation of the response: We can also represent $u = u(t)$ by

$$u(t) = G_1\exp(i\omega_n t) + G_2\exp(\omega_n t) \tag{15}$$

$$= (G_{1R} + iG_{1I})\left(\cos(\omega_n t) + i\sin(\omega_n t)\right)$$
$$+ (G_{2R} + iG_{2I})\left(\cos(\omega_n t) - i\sin(\omega_n t)\right) \tag{16}$$

$$= (G_R + iG_I)\exp(i\omega_n t) + (G_R - iG_I)\exp(-i\omega_n t) \tag{17}$$

With the initial conditions and Euler's formula ($e^{\pm ix} = \exp(\pm ix) = \cos x \pm i\sin x$), we have

$$\boxed{u(t) = |u|\cos(\omega_n t + \theta)} \tag{18}$$

$$\theta = \tan^{-1}\left[\frac{-\dot{u}_0}{\omega_n u_0}\right] \tag{19}$$

- ## Damped SDOF systems

  - Damped: Damping force is considered in the SDOF system.
  - Solution to the free vibration problem (ODE) of damped SDOF systems is a particular solution to the ODE.

– Governing equation of a damped SDOF mass-damper-spring system:

$$m\ddot{u} + c\dot{u} + ku = 0 \tag{20}$$

Solution to Eq.(20) depends on the relation between c and $2m\omega_n$. Therefore, a ratio is defined to categorize different responses of the SDOF system.

$$\boxed{\xi = \frac{c}{c_c} = \frac{c}{2m\omega_n}} \tag{21}$$

is the damping ratio of the SDOF system. When $\xi = 1$ or $c_c = 2m\omega$, the SDOF system has critical damping or is a critically-damped system.

- **Underdamped SDOF systems**
  – When $\xi < 1$ or $c < 2m\omega_n$, the SDOF system is called undercritically-damped or underdamped. The solution to Eq.(20) is

$$\boxed{u(t) = \exp(-\xi\omega_n t)\left[u_0 \cos(\omega_D t) + \frac{\dot{u}_0 + \xi\omega_n u_0}{\omega_n}\sin(\omega_D t)\right]} \tag{22}$$

The natural frequency of this underdamped SDOF system is

$$\boxed{\omega_D = \omega_n\sqrt{1-\xi^2}} \tag{23}$$

- **Critically-damped SDOF systems**
  – When $\xi = 1$ or $c = 2m\omega_n = c_c$, the SDOF system is called critically-damped. The solution to Eq.(20) is

$$\boxed{u(t) = \exp(-\omega_n t)\left[u_0\left(1 - \omega_n t\right) + \dot{u}_0 t\right]} \tag{24}$$

The free vibration of critically-damped SDOF systems has no oscillation. Hence, no natural frequency is defined.

- **Overdamped SDOF systems**
  – When $\xi > 1$ or $c > 2m\omega_n$, the SDOF system is called overcritically-damped or overdamped. The solution to Eq.(20) is

$$\boxed{u(t) = \exp(-\omega_n t)\left[A\sinh(\hat{\omega}t) + B\cosh(\hat{\omega}t)\right]} \tag{25}$$

$$A = \frac{\dot{u}_0 + (\xi + \sqrt{\xi^2 - 1})\omega_n u_0}{2\omega_n\sqrt{\xi^2-1}} \tag{26}$$

$$B = \frac{-\dot{u}_0 - (\xi + \sqrt{\xi^2 - 1})\omega_n u_0}{2\omega_n\sqrt{\xi^2-1}} \tag{27}$$

where

$$\boxed{\hat{\omega} = \omega_n \sqrt{\xi^2 - 1}} \tag{28}$$

is the natural frequency of overdamped SDOF systems in free vibration. Also,

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = \frac{\exp(x) - \exp(-x)}{2} \tag{29}$$

$$\cosh \frac{e^x + e^{-x}}{2} = \frac{\exp(x) + \exp(-x)}{2} \tag{30}$$

- Example on an underdamped SDOF portal frame

Reading

[AKC: Ch02]

References

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