Forced Vibration of Single-Degree-of-Freedom (SDOF) Systems

- **Dynamic response of SDOF systems subjected to external loading**

  - **Governing equation of motion** –

$$m\ddot{u} + c\dot{u} + ku = P(t) \tag{1}$$

the complete solution is

$$\boxed{u = u_{\text{homogeneous}} + u_{\text{particular}} = u_{\text{h}} + u_{\text{p}}} \tag{2}$$

where u_{h} is the homogeneous solution to the PDE or the free vibration response for $P(t) = 0$, and u_{p} is the particular solution to the PDE or the response for $P(t) \neq 0$.

  - **Types of motion (displacement)** –
    1. Undamped systems ($c = 0, \xi = 0$) – Oscillation
    2. Undercritically-damped or underdamped systems ($c < c_c, \xi < 1$) – Oscillation (in general), also depending on I.C.
    3. Critically-damped systems ($c = c_c, \xi = 1$) – Decaying response with at most one reversal, depending on I.C.
    4. Overcritically-damped or overdamped systems ($c > c_c, \xi > 1$) – Decaying response, depending on I.C.
       * Case 1: Displacement never crosses the axis (non-reversal)

$$\left|\frac{\dot{u}_0 + \xi\omega_n u_0}{\omega_D u_0}\right| < 1 \tag{3}$$

       * Case 2: Displacement crosses the axis once

$$\left|\frac{\dot{u}_0 + \xi\omega_n u_0}{\omega_D u_0}\right| > 1 \tag{4}$$

where the critical damping coefficient $c_c = 2\sqrt{km}$ (or c_{cr}) is the smallest value of c that exhibits oscillation completely. (See Figure 1.)

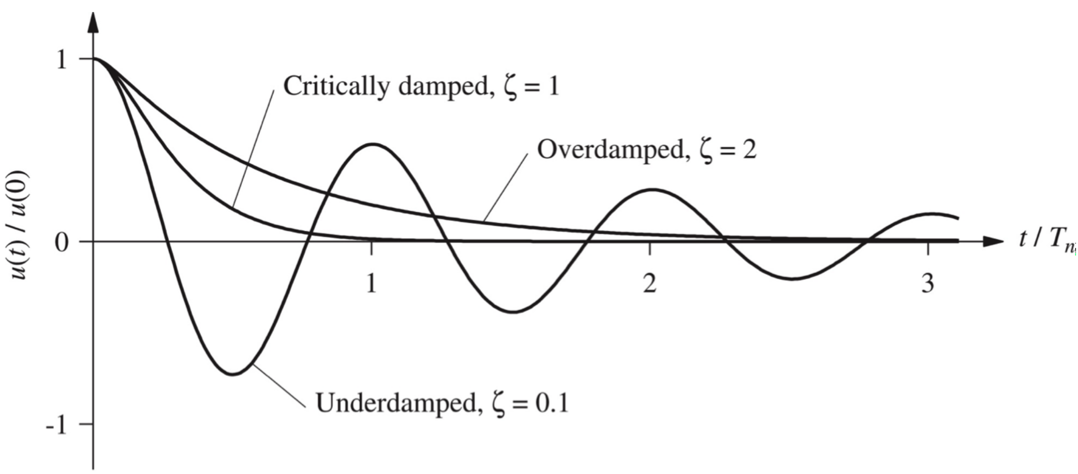

Figure 1: Free vibration displacement response ratio of SDOF systems

- **Particular solutions to the PDE with special loading functions**
  - **Impulse/Dirac delta function** –

The response of a SDOF system subjected to a unit impulse force having a finite time integral can be determined by the time integral for the force.

$$\hat{P} = \int P(t)dt \tag{5}$$

where $\hat{P}$ is the linear impulse of force $P(t)$. When the duration of $P(t)$ approaches zero ($t \to 0$), the impulse force approaches infinity but $\hat{P}$ becomes equal to unity or the unit impulse. This unit impulse is also known as the *Dirac delta function* defined by

$$\delta\left(t-\tau\right) = 0 \tag{6}$$

for $t \neq \tau$ and

$$\int_0^{\infty} \delta\left(t\right) dt = 1 \tag{7}$$

$$\int_0^{\infty} \delta\left(t-\tau\right) P(t)dt = P(\tau) \tag{8}$$

in which $0 < \tau < \infty$. Therefore, an impulse force $P(t)$ acting at $t = \tau$ can be represented as

$$P(t) = \hat{P}\delta\left(t-\tau\right) \tag{9}$$

Mathematically, $t \to 0$ is considered as $t = t^+$ (in real time). Eq.(8) becomes

$$\int_0^{0^+} P(t)dt = \int_0^{0^+} \hat{P}\delta(t)dt = m\dot{u}\left(0^+\right) \tag{10}$$

which is the impulse-momentum theorem. The initial velocity is

$$\dot{u}\left(0^+\right) = \frac{\hat{P}}{m} \tag{11}$$

The transient or free vibration displacement response for a SDOF system subjected to initial velocity becomes

$$u(t) = \frac{\hat{P}}{m\omega_D}\sin\left(\omega_D t\right)\exp\left(-\xi\omega_n t\right) \tag{12}$$

where $\omega_D = \omega_n\sqrt{1-\xi^2}$. Should we define

$$u(t) = \hat{P}h(t), \tag{13}$$

$h(t)$ is the impulse response function and

$$h(t) = \frac{1}{m\omega_D}\sin\left(\omega_D t\right)\exp\left(-\xi\omega_n t\right) \tag{14}$$

for damped SDOF systems. For undamped SDOF systems,

$$h(t) = \frac{1}{m\omega_n}\sin\left(\omega_n t\right) \tag{15}$$

– **Duhamel's integral** –

The response of a SDOF system to arbitrary forms of excitation can be analyzed with the aid of the impulse function $h(t)$ with magnitude of $P(\tau)$. To do so, the arbitrary excitation $P(t)$ is considered consisting of a sequence of impulse forces $P(\tau)$ acting over a very small time interval $d\tau$. The displacement response to each impulse is valid for all time $t > \tau$. Therefore, the incremental response du to each impulse $P(\tau)$ can be expressed as

$$du = P(\tau)d\tau h\left(t-\tau\right) \tag{16}$$

The total response to $P(t)$ is obtained by superimposing / integrating the individual incremental responses du due to each impulse over the duration of loading.

$$u_p(t) = \int_0^t P(\tau)h(t-\tau)d\tau = P * h \tag{17}$$

Eq.(17) is known as *Duhamel's integral* or the *convolution integral*, which is only applicable to linear systems. (Q: Why?) $*$ is the convolution symbol. Recall Eq.(14) for damped SDOF systems,

$$u_p(t) = \frac{1}{m\omega_D}\int_0^t P(\tau)\sin\left[\omega_D\left(t-\tau\right)\right]\exp\left[-\xi\omega_n\left(t-\tau\right)\right]d\tau \tag{18}$$

The complete solution for impulse-loaded, undercritically-damped SDOF systems is obtained.

$$\begin{aligned} u(t) &= u_h(t) + u_p(t) \\ &= \left[u_0\cos\omega_D t + \left(\frac{\dot{u_0} + u_0\xi\omega_n}{\omega_D}\right)\sin\omega_D t\right]\cdot\exp\left(-\xi\omega_n t\right) \\ &\quad + \frac{1}{m\omega_D}\int_0^t P(\tau)\sin\left[\omega_D\left(t-\tau\right)\right]\exp\left[-\xi\omega_n\left(t-\tau\right)\right]d\tau \end{aligned} \tag{19}$$

– **Step load of infinite duration** –

$$P(t) = \begin{cases} 0 & : \quad t < 0 \\ P_0 & : \quad t \geq 0 \end{cases} \tag{20}$$

– **Forcing function is polynomial in time** –

$$P(t) = a + bt + ct^2 + ... \tag{21}$$

where $a, b, c, ...$ are constants in defining the forcing function.

– **Step load of finite duration** –

$$P(t) = \begin{cases} 0 & : \quad t < 0 \\ P_0 & : \quad 0 < t \leq t_d \\ 0 & : \quad t_d < t \end{cases} \tag{22}$$

• Harmonic vibration of SDOF systems

– **Undamped SDOF systems** –

$$m\ddot{u} + ku = P(t) = P_0 \sin(\omega_p t) \tag{23}$$

where ω_p is the loading frequency. The particular solution is

$$u_p(t) = \frac{P_0}{k} \cdot \frac{1}{1 - \left(\frac{\omega_p}{\omega_n}\right)^2} \sin(\omega_p t) \tag{24}$$

In harmonic vibration, due to the nature of external loading $P(t)$, the complete solution represents the sum of two states:

$$u(t) = u_h(t) + u_p(t) \tag{25}$$

where $u_h(t)$ represents the *transient state* response and $u_p(t)$ the *steady state* response. (See Figure 2.) The maximum static displacement is

$$(u_{st})_0 = \frac{P_0}{k} \tag{26}$$

For $\frac{\omega_p}{\omega_n} < 1$ or $\omega_p < \omega_n$, $\frac{1}{1 - \left(\frac{\omega_p}{\omega_n}\right)^2}$ is positive, suggesting that $u_p(t)$ and $P(t)$ have same algebraic sign; the displacement is **in**

phase with the applied force. On the other hand, when $\dfrac{\omega_p}{\omega_n} > 1$ or $\omega_p > \omega_n$, $\dfrac{1}{1-\left(\dfrac{\omega_p}{\omega_n}\right)^2}$ is negative and the displacement is **out of phase** with the applied force.

– **Damped SDOF systems** –

$$m\ddot{u} + c\dot{u} + ku = P(t) = P_0 \sin(\omega_p t) \tag{27}$$

where ω_p is the loading frequency. The particular solution is

$$u_p(t) = C\sin(\omega_p t) + D\cos(\omega_p t) \tag{28}$$

where

$$C = \frac{P_0}{k} \cdot \frac{1-\left(\dfrac{\omega_p}{\omega_n}\right)^2}{\left[1-\left(\dfrac{\omega_p}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\dfrac{\omega_p}{\omega_n}\right)\right]^2}$$

$$D = \frac{P_0}{k} \cdot \frac{-2\xi\dfrac{\omega_p}{\omega_n}}{\left[1-\left(\dfrac{\omega_p}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\dfrac{\omega_p}{\omega_n}\right)\right]^2} \tag{29}$$

The complete solution is still

$$u(t) = u_h(t) + u_p(t) \tag{30}$$

(See Figure 3.)

- ## Dynamic response factors

– **Undamped SDOF systems** –

Eq.(24) can be written as

$$u_p(t) = (u_{st})_0 \, R_d \sin(\omega_p t - \phi) \tag{31}$$

where the displacement response factor R_d and the phase angle (or phase lag) ϕ are

$$R_d = \frac{u_0}{(u_{st})_0} = \frac{1}{\left|1-\left(\dfrac{\omega_p}{\omega_n}\right)^2\right|} \tag{32}$$

$$\phi = \begin{cases} 0^\circ & : \quad \omega_p < \omega_n \\ 180^\circ & : \quad \omega_p > \omega_n \end{cases} \tag{33}$$

In the case of $\omega_p = \omega_n$, $R_d \to \infty$; ω_p is the *resonant frequency* of the undamped SDOF system. However, resonance does not immediately result in excessive R_d but only gradually lead to excessive R_d. Figure 4 shows the displacement ratio of an undamped SDOF system to sinusoidal force in resonance (loading frequency = natural frequency). Note the difference between Figure 2 and Figure 4. They are both harmonic response of an undamped SDOF system, except one is NOT in resonance (Figure 2) and the other is (4).

– **Damped SDOF systems** –
The displacement response factor R_d and the phase angle ϕ for damped SDOF systems is

$$R_d = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega_p}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega_p}{\omega_n}\right)\right]^2}} \tag{34}$$

$$\phi = \tan^{-1}\left[\frac{2\xi\left(\frac{\omega_p}{\omega_n}\right)}{1 - \left(\frac{\omega_p}{\omega_n}\right)^2}\right] \tag{35}$$

When $\frac{\omega_p}{\omega_n} \ll 1$, R_d is independent of damping and

$$u_0 \cong (u_{st})_0 = \frac{P_0}{k} \tag{36}$$

which is the static deformation of the SDOF system. The dynamic response of this kind is controlled by the stiffness of the system. When the $\frac{\omega_p}{\omega_n} \gg 1$, R_d approaches zero as $\frac{\omega_p}{\omega_n}$ increases and is unaffected by damping. If $\omega_p = \omega_n$,

$$u_0 = \frac{(u_{st})_0}{2\xi} = \frac{P_0}{c\omega_n} \tag{37}$$

And the phase angle ϕ is

$$\phi = \begin{cases} 0^\circ & : \quad \omega_p \ll \omega_n \\ 90^\circ & : \quad \omega_p = \omega_n \\ 180^\circ & : \quad \omega_p \gg \omega_n \end{cases} \tag{38}$$

The velocity response factor R_v is

$$R_v = \frac{\omega_p}{\omega_n} R_d \tag{39}$$

and the acceleration response factor R_a is

$$R_a = \left(\frac{\omega_p}{\omega_n}\right)^2 R_d \tag{40}$$

Or equivalently,

$$\frac{R_a}{\frac{\omega_p}{\omega_n}} = R_v = \frac{\omega_p}{\omega_n} R_d \tag{41}$$

Figure 5 shows the frequency-response curves of the deformation response factor R_d for a few values of ξ. Velocity and acceleration response factor curves are shown in Figure 6.

- **Natural frequency and damping from harmonic tests**
  - Resonance testing

$$\xi = \frac{1}{2} \frac{(U_{st})_0}{(U_0)_{\omega=\omega_n}} \tag{42}$$

  - Frequency-response curve

$$\xi = \frac{\omega_b - \omega_a}{2\omega_n} = \frac{f_b - f_a}{2f_n} \tag{43}$$

- **Force transmission and vibration isolation**
  - Transmissibility (TR): Figure 8 shows the transmissibility curves of various SDOF systems with different levels of damping.
- **Example – Transmissibility**

Remark

All figures in this lecture note are from Prof. Chopra's book, *Dynamics of Structures*.

Reading

[AKC: Ch03 – 3.1, 3.2, Ch04 – 4.1∼4.7]

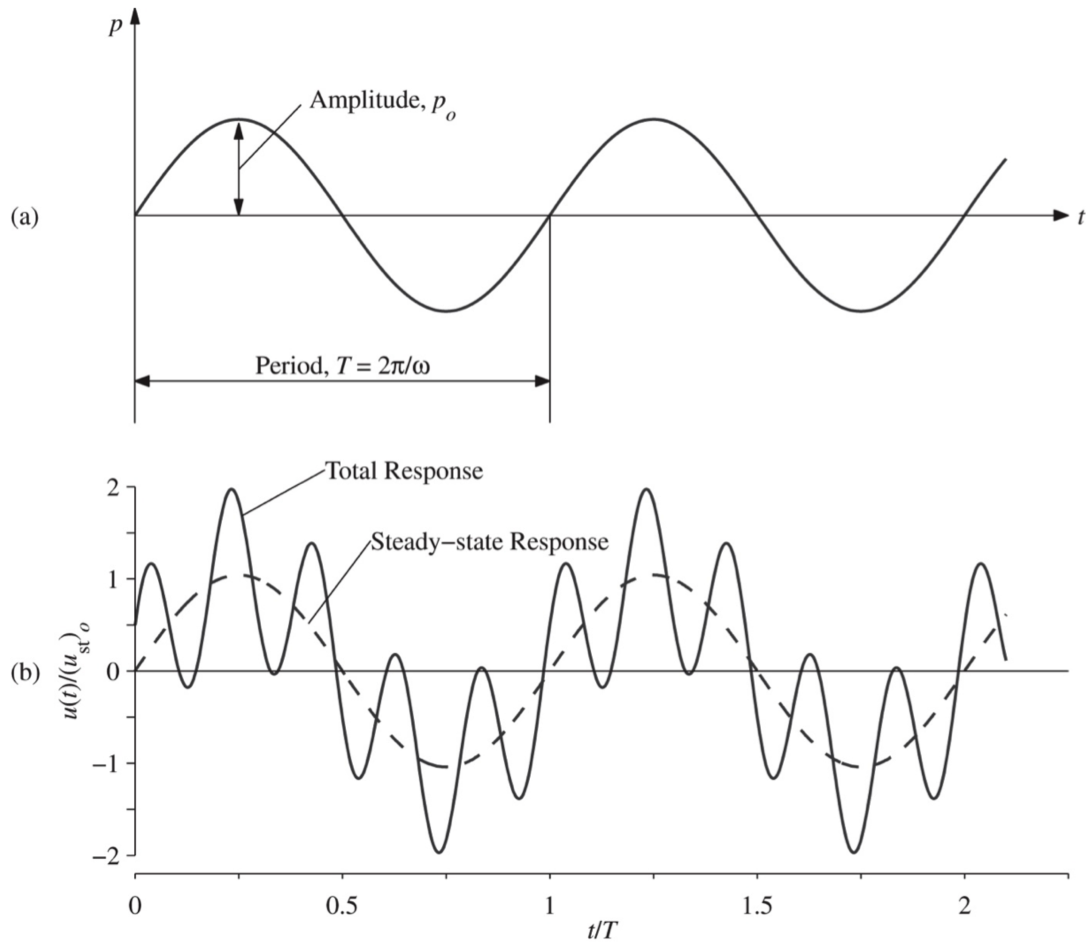

Figure 2: Harmonic force and the displacement response of an undamped SDOF system ; $\dfrac{\omega}{\omega_n} = 0.2$, $u_0 = \dfrac{p_0}{2k}$

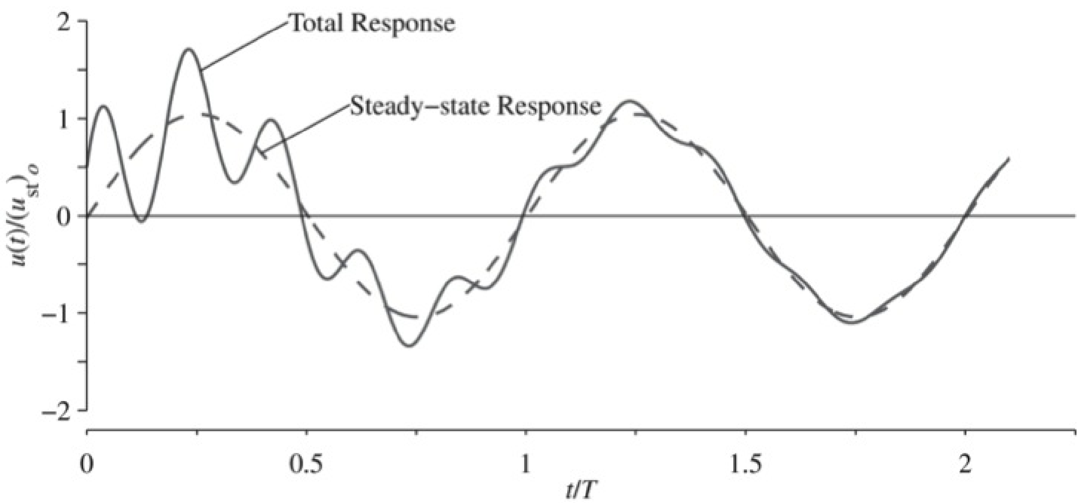

Figure 3: Displacement response of an undercritically-damped SDOF system; $\dfrac{\omega}{\omega_n} = 0.2$, $\xi = 0.05$, $u_0 = \dfrac{p_0}{2k}$

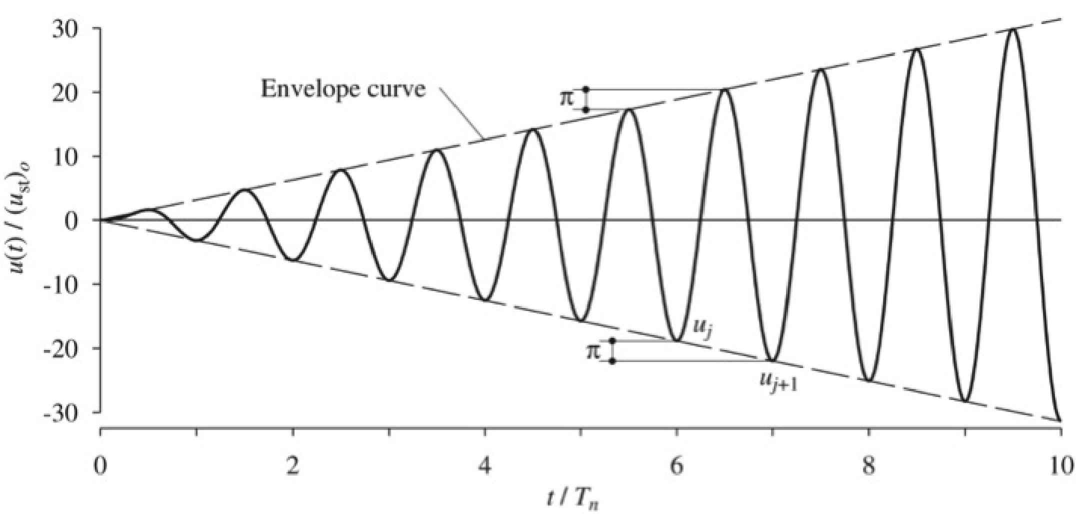

Figure 4: Displacement response ratio of an undamped SDOF system; $\dfrac{\omega}{\omega_n} = 1$, $\xi = 0$, $u_0 = \dot{u}_0 = 0$,

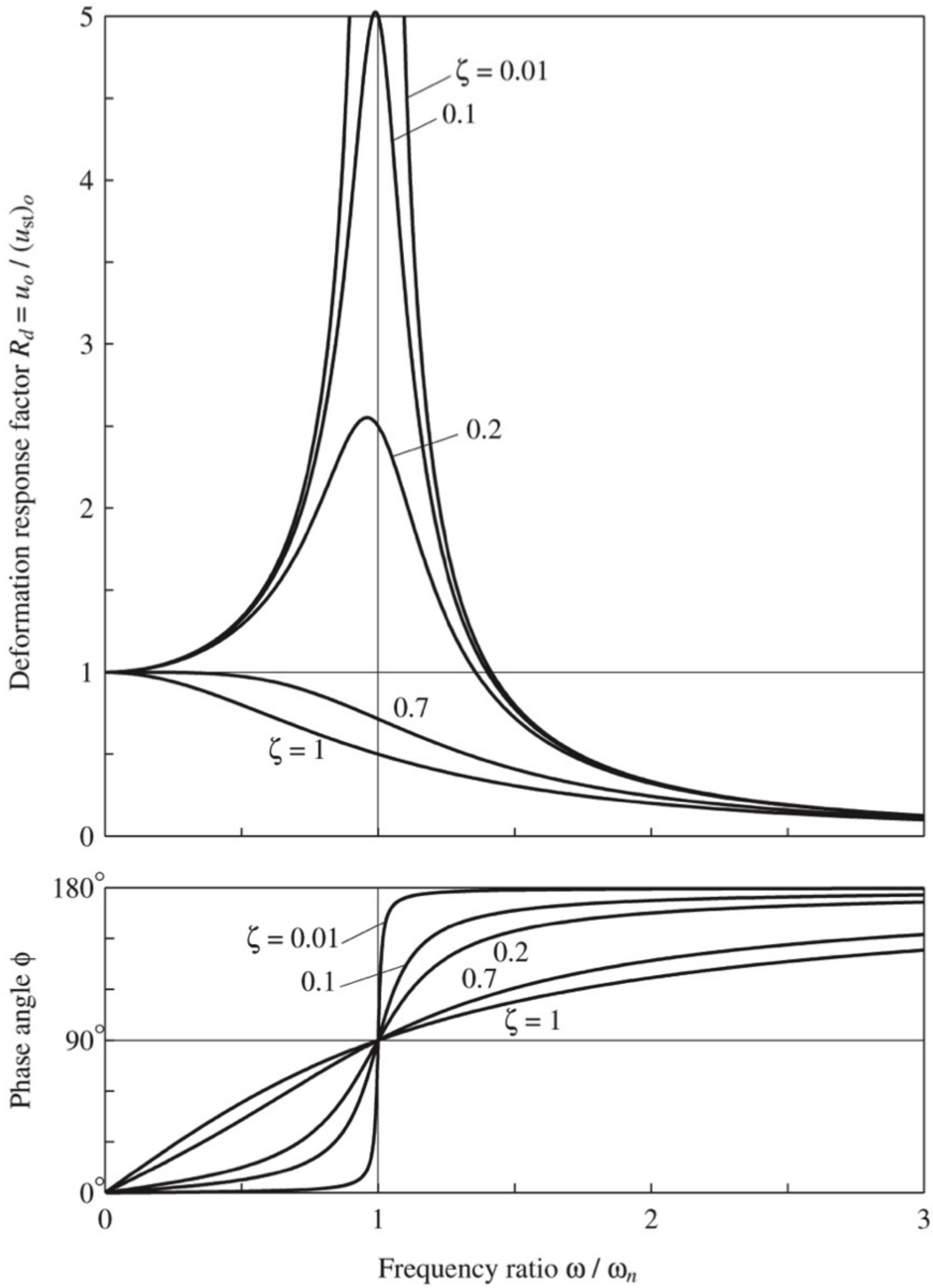

Figure 5: Deformation response factor and phase angle for a damped system excited by harmonic force

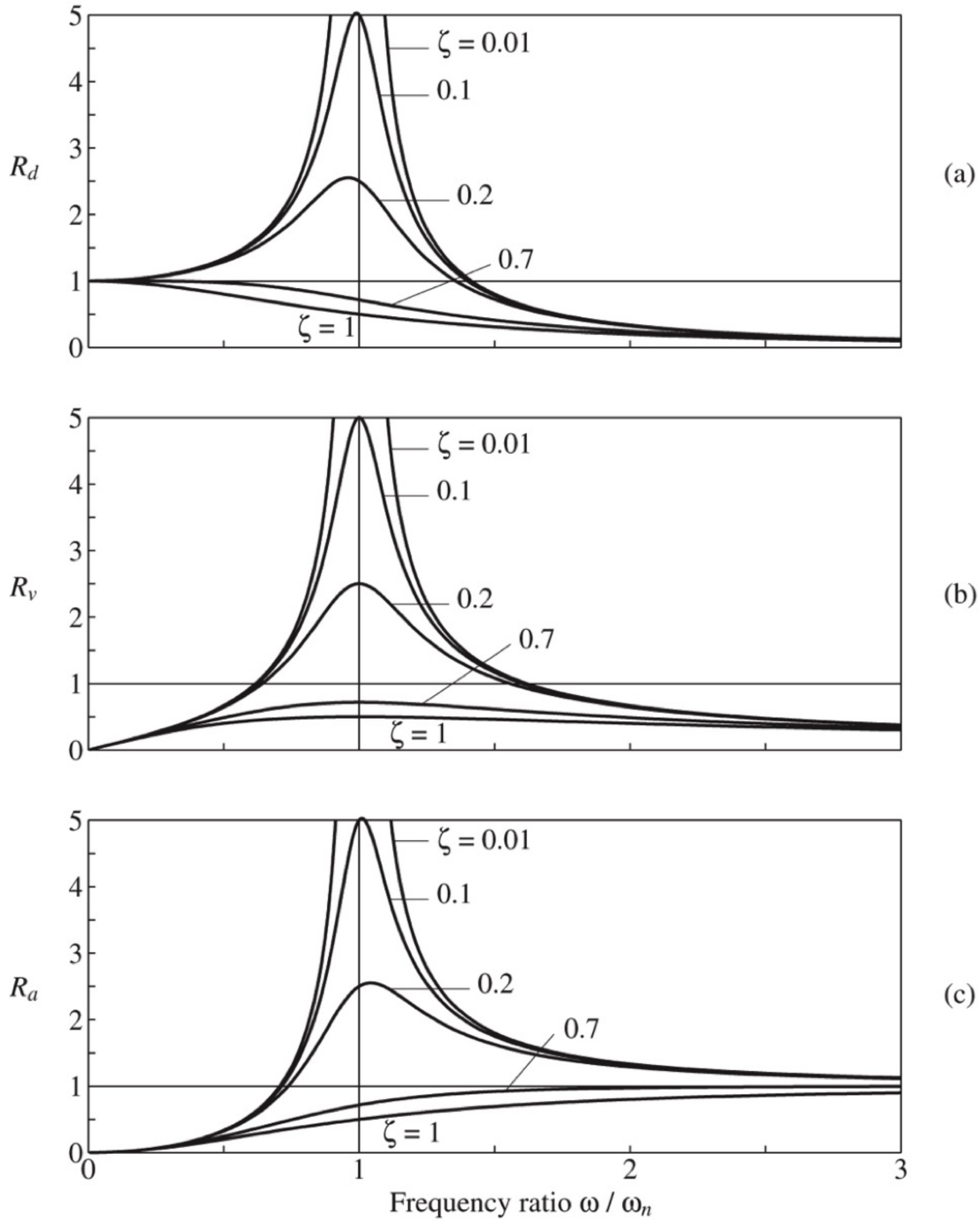

Figure 6: Deformation, velocity, and acceleration response factors for a damped system excited by harmonic force

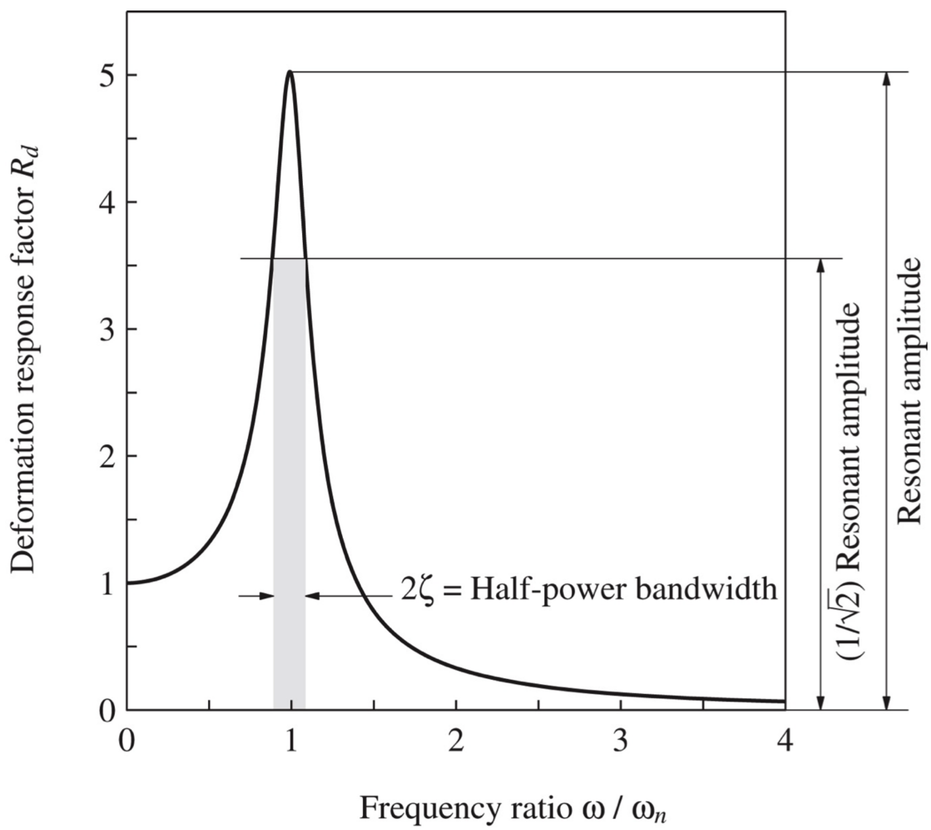

Figure 7: Definition of half-power bandwidth

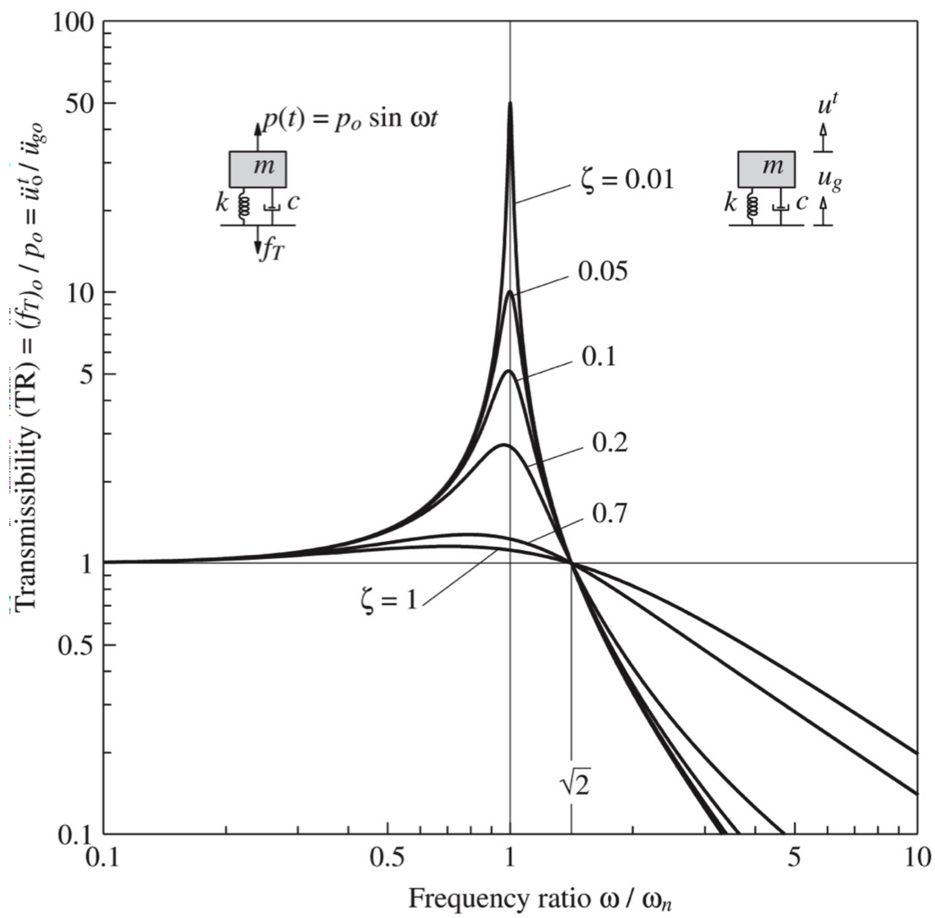

Figure 8: Transmissibility of harmonic excitation