Free Vibration of Multi-Degree-of-Freedom (MDOF) Systems

- **General**
  - Static DOF (total number of independent deformation modes) versus dynamic DOF (total number of local inertial forces)
  - MDOF systems have multiple modal frequencies and mode shapes.
  - Both SDOF and MDOF systems are models simplified from real structures.
  - In MDOF systems, dynamic equilibrium is attained locally for each DOF and globally for the system. The system's dynamic governing equation can be derived when all DOFs are evaluated.
  - Discrete MDOF systems are usually considered in practical situations due to the ease of numerical implementation; linear algebra and matrix calculus are used.
  - Consideration of boundary condition (B.C.)
- **Determination of structural properties for discrete MDOF systems**
  - **Mass** – Mass lumping (types of inertial forces; linear and rotational)
  - **Damping** – Usually associated with individual DOF's mass and stiffness, e.g., viscous damping.
  - **Stiffness** – Flexibility approach; $F = K^{-1}$.
  - **Example – Mass and Stiffness Matrices of Building Models**
    - **When $EI_b \approx \mathbf{0}$ –**

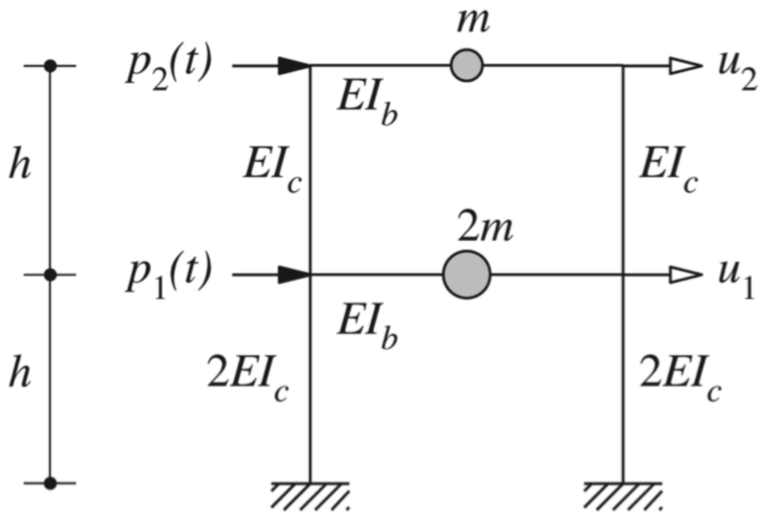

Figure 1: Mass and stiffness matrices of a two-story building model

* **When** $EI_b = EI_c$ –

* **When** $EI_b \to \infty$ –

– **Example – Stiffness Matrix of a Cantilever Beam**

- **Approaches for deriving dynamic governing equations**

  1. **Dynamic equilibrium** – Newton's second law of motion and D'Alembert's principle
  2. **Lagrange's equation of motion**
  3. **Hamilton's principle**

  Lagrange's and Hamilton's approaches are not easy to be implemented for numerical computation; subroutines for differentiation and integration are required.

- **Formation of governing equations with active DOFs**

  – Static condensation

    * The mass matrix formulation usually is associated with translational DOF only, while the formulation of the stiffness matrix is associated with both translational and rotational DOFs, leading to a large stiffness matrix.
    * The purpose of static condensation is to eliminate the extraneous DOFs associated with rotation from the stiffness matrix before the equations of motion can be written.

  Consider the stiffness matrix to be of the following form.

$$\begin{bmatrix} K_{tt} & K_{t\theta} \\ K_{\theta t} & K_{\theta\theta} \end{bmatrix} \begin{bmatrix} u \\ \theta \end{bmatrix} = \begin{bmatrix} P_t \\ P_\theta \end{bmatrix} = \begin{bmatrix} P_t \\ 0 \end{bmatrix} \tag{1}$$

where u and θ are the subvectors of the displacements to be retained (translational) and condensed out (rotational). When there is no rotational force subvectors acting in the structure, it is possible to express the rotational displacement by means of the following form.

$$\theta = -K_{\theta\theta}^{-1} K_{\theta t} u \tag{2}$$

From Eq.(1), we also have

$$K_{tt} u + K_{t\theta} \theta = P_t \tag{3}$$

Substituting Eq.(2) into Eq.(3) gives

$$K_{tt} u - K_{t\theta} K_{\theta\theta}^{-1} K_{\theta t} u = P_t \tag{4}$$

Since the purpose of static condensation is to obtain

$$K_t u = P_t \tag{5}$$

we now find the K_t to be

$$\boxed{K_t = K_{tt} - K_{t\theta} K_{\theta\theta}^{-1} K_{\theta t}} \tag{6}$$

as the statically condensed stiffness matrix which contains translational terms only. Note that in K_t, $K_t t$ contains longitudinal and transverse stiffness terms.

- Kinematic condensation
  - In some applications such as buildings and bridges, transverse stiffness terms are of major concerned.
  - The purpose of kinematic condensation is to eliminate the DOFs that are of secondary importance in the response of structures.

The relationship between K_{tt} (contains both longitudinal and transverse stiffness terms) and $\bar{K}_{tt}$ (contains only longitudinal stiffness terms) can be defined by a transformation matrix T.

$$K_{tt} = T \bar{K}_{tt} \tag{7}$$

$$\bar{K}_{tt} = T^T K_{tt} \tag{8}$$

For example,

$$K_{tt} = T\bar{K}_{tt}$$

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{9}$$

Or

$$\bar{K}_{tt} = T^T K_{tt}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^T \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} \tag{10}$$

Similarly, T can be applied to other terms in the equation of motion.

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{P} \tag{11}$$

Replacing $\ddot{\mathbf{u}}$, $\dot{\mathbf{u}}$, and $\mathbf{u}$ with $T\ddot{\bar{\mathbf{u}}}$, $T\dot{\bar{\mathbf{u}}}$, and $T\bar{\mathbf{u}}$ in Eq.(11) gives

$$\mathbf{m}T\ddot{\bar{\mathbf{u}}} + \mathbf{c}T\dot{\bar{\mathbf{u}}} + \mathbf{k}T\bar{\mathbf{u}} = \mathbf{P} \tag{12}$$

Pre-multiply T^T to each term in Eq.(12).

$$\begin{aligned} T^T\mathbf{m}T\ddot{\bar{\mathbf{u}}} + T^T\mathbf{c}T\dot{\bar{\mathbf{u}}} + T^T\mathbf{k}T\bar{\mathbf{u}} &= T^T\mathbf{P} \\ \bar{\mathbf{m}}\ddot{\bar{\mathbf{u}}} + \bar{\mathbf{c}}\dot{\bar{\mathbf{u}}} + \bar{\mathbf{k}}\bar{\mathbf{u}} &= \bar{\mathbf{P}} \end{aligned} \tag{13}$$

where $\bar{\mathbf{m}} = T^T\mathbf{m}T$, $\bar{\mathbf{c}} = T^T\mathbf{c}T$, and $\bar{\mathbf{k}} = T^T\mathbf{k}T$ are the kinematically condensed mass, damping, and stiffness matrices, respectively.

- ## Concept of mode shapes and modal frequencies

  Consider an undamped MDOF system in free vibration. The governing equation is

  $$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = 0 \tag{14}$$

  where $\mathbf{m}$ is the mass matrix, $\ddot{u}$ is the acceleration vector, $\mathbf{k}$ is the stiffness matrix, and $\mathbf{u}$ is the displacement vector, both of the n-th mode. Note that $\mathbf{u} = \mathbf{u}(x,t)$. The solution is

  $$\mathbf{u}(x,t) = q_n(t)\phi_n(x) \tag{15}$$

where $q_n(t)$ is the generalized coordinate (time-independent) and $\phi_n(x)$ is the shape function or mode shape. The governing equation becomes

$$\left[\mathbf{k} - \omega_n^2 \mathbf{m}\right] \phi_n = 0 \tag{16}$$

where ω_n is the modal frequency of the n-th mode and is evaluated by solving the following equation.

$$\det \left[\mathbf{k} - \omega_n^2 \mathbf{m}\right] = 0 \tag{17}$$

(Example of a building-type model)

- **Orthogonality of modes**
  - Property I: $\phi_i^T \phi_j = 0$ when $i \neq j$
  - Property II: $\phi_i^T \phi_j = 1$ when $i = j$

leading to

- $\phi_i^T \mathbf{k} \phi_j = 0$ and $\phi_i^T \mathbf{m} \phi_j = 0$ and $\phi_i^T \mathbf{c} \phi_j = 0$ when $i \neq j$
- $\phi_i^T \mathbf{k} \phi_j = k_i$ and $\phi_i^T \mathbf{m} \phi_j = m_i$ and $\phi_i^T \mathbf{c} \phi_j = c_i$ when $i = j$

The mode shape matrix (or modal matrix) is formed by

$$\mathbf{\Phi} = [\phi_{jn}] \tag{18}$$

where n is the number of modes. Therefore, the diagonal property matrices are

$$\begin{aligned} \mathbf{M} &= \mathbf{\Phi}^T \mathbf{m} \mathbf{\Phi} \\ \mathbf{C} &= \mathbf{\Phi}^T \mathbf{c} \mathbf{\Phi} \\ \mathbf{K} &= \mathbf{\Phi}^T \mathbf{k} \mathbf{\Phi} \end{aligned} \tag{19}$$

Mode shapes can be normalized by setting its largest element to be unity. If normalized to individual DOF's mass, we have

$$M_n = \phi_n^T \mathbf{m} \phi_n = 1$$
$$\mathbf{\Phi}^T \mathbf{m} \mathbf{\Phi} = \mathbf{I} \tag{20}$$

where $\mathbf{I}$ is the identity matrix.

- **Modal expansion of displacements**

$$\mathbf{u} = \sum_{i=1}^{n} \phi_i q_i = \mathbf{\Phi q} \tag{21}$$

where $\mathbf{q}$ is the modal coordinates or normal coordinates which are determined by

$$q_i = \frac{\phi_i^T \mathbf{mu}}{\phi_i^T \mathbf{m} \phi_i} = \frac{\phi_i^T \mathbf{mu}}{M_n} = \phi_i^T \mathbf{mu} \tag{22}$$

if normalized w.r.t. mass.

- **Estimation of modal frequencies using Rayleigh's quotient**

$$\lambda = \frac{\phi^T \mathbf{k} \phi}{\phi^T \mathbf{m} \phi} \tag{23}$$

where λ is called Rayleigh's quotient.

  – When ϕ is the actual mode shape vector ϕ_i of the i-th mode, $\lambda = \omega_i^2$.

  – When ϕ is a first-order approximation to the actual mode shape vector, the estimated modal frequency is with an second-order error.

  – λ is bounded between $\lambda_1 = \omega_1^2$ and $\lambda_n = \omega_n^2$.

- **Estimation of modal frequencies using inverse iterative method**

  1. Choose a starting vector $\mathbf{v_j}$.

2. Determine $\mathbf{v_{j+1}}$ by solving

$$\mathbf{k}\mathbf{v_{j+1}} = \mathbf{m}\mathbf{v_j} \tag{24}$$

3. Compute the norm of $\mathbf{v_{j+1}}$ by

$$a_{j+1} = \sqrt{\mathbf{v_{j+1}^T}\mathbf{v_j}} \tag{25}$$

4. Normalize $\mathbf{v_{j+1}}$ w.r.t. the mass matrix.

$$\mathbf{v_{j+2}} = \frac{\sqrt{\mathbf{v_{j+1}^T}\mathbf{v_j}}}{\sqrt{\mathbf{v_{j+1}^T}\mathbf{m}\mathbf{v_j}}} \tag{26}$$

5. $\mathbf{v_{j+2}}$ converges if it satisfies

$$\mathbf{k}\mathbf{v_{j+2}} = \mathbf{m}\mathbf{v_{j+1}} \tag{27}$$

   where the ratio between any two vectors converges to the square of the fundamental frequency ω_1^2.

$$\mathbf{v_{j+1}} \approx \frac{1}{\omega_1^2}\mathbf{v_j} \tag{28}$$

6. If $\mathbf{v_{j+2}}$ does not converge, repeat steps 2 5.

Reading

[AKC: Ch9, Ch10]

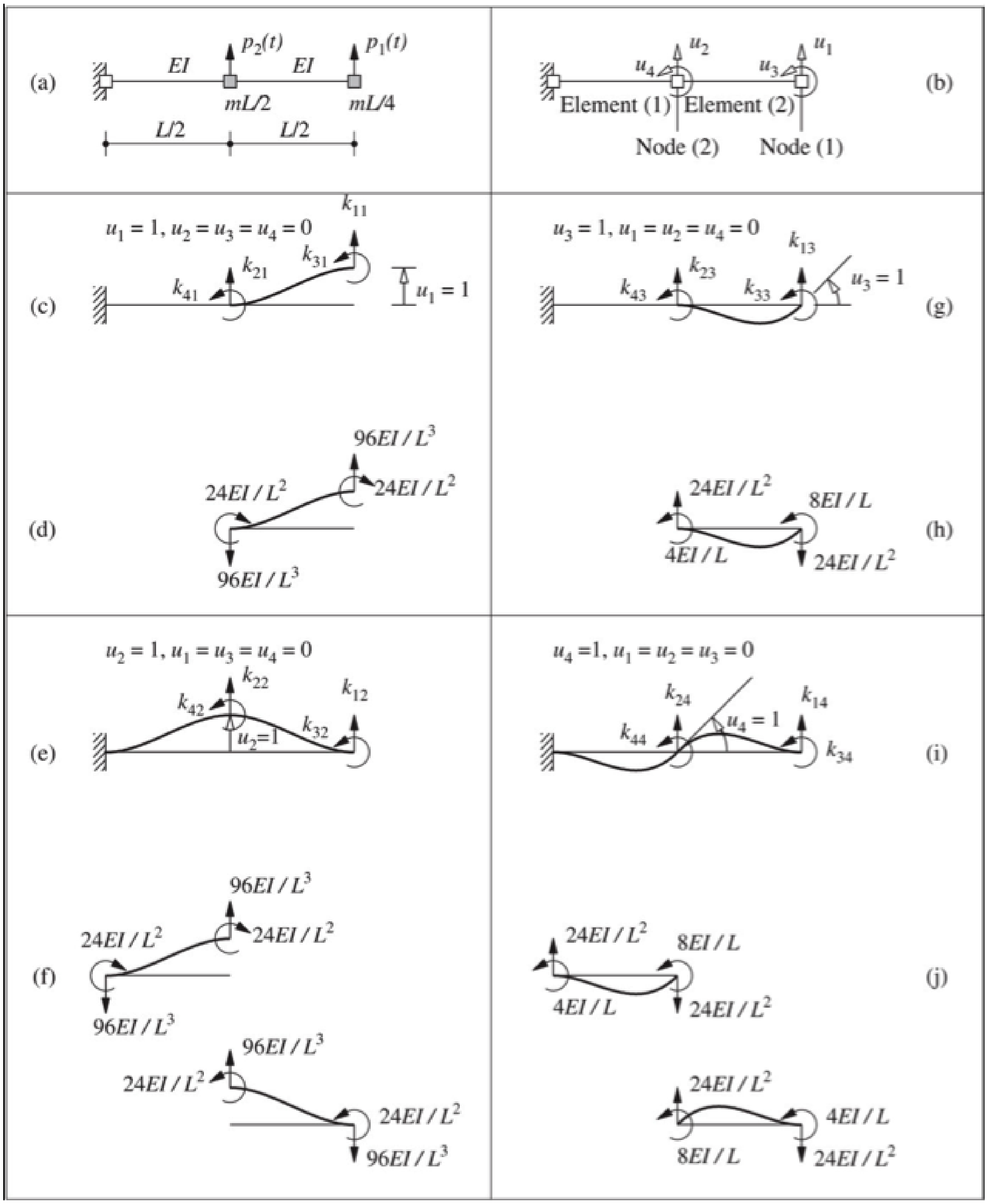

Figure 2: Stiffness matrix of a cantilever beam