Tuned Mass Damper (TMD) Systems

- **General**
  - TMD is "a device consisting of a mass, a spring, and a damper that is attached to a structure in order to reduce the dynamic response of the structure", which is a concept first introduced by H. Frahm (1909).
  - The purpose of using TMD is to produce an artificial force originated from the mass of TMD in order to counteract the structural vibration at which the TMD is installed.
  - TMD is most effective for periodic excitations and wind loads.

- **Types of TMD**

Figure 1 provides several possible types of TMD for buildings.

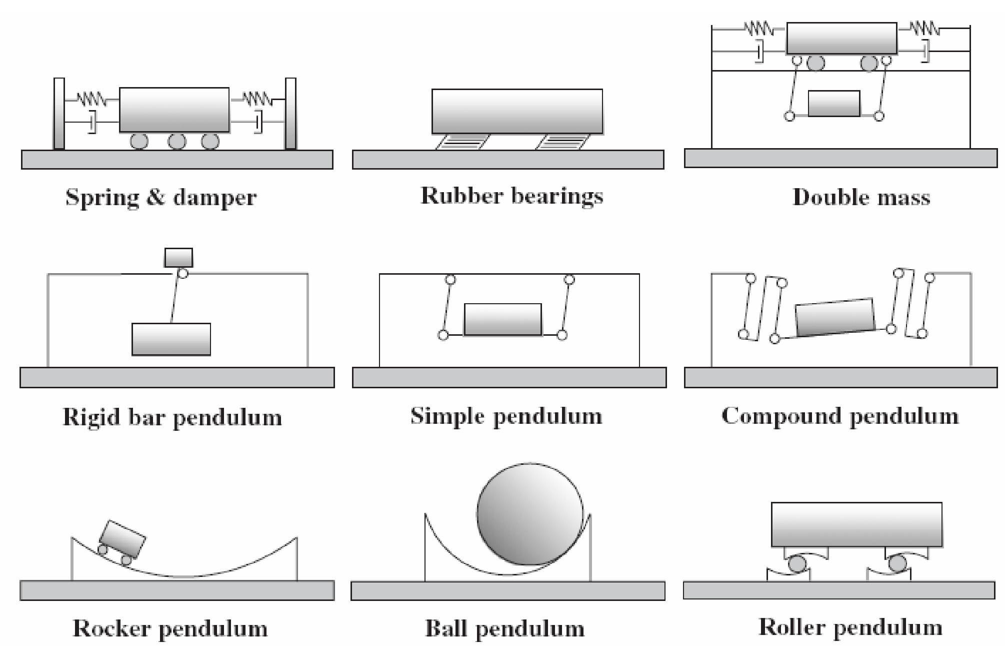


Figure 1: Types of TMD for buildings [Matta and De Stefano (2009)]

- **TMD Theory for SDOF Systems**

Generally, a SDOF-TMD system can be modeled as shown in Figure 2. Various cases of SDOF-TMD systems can be defined by

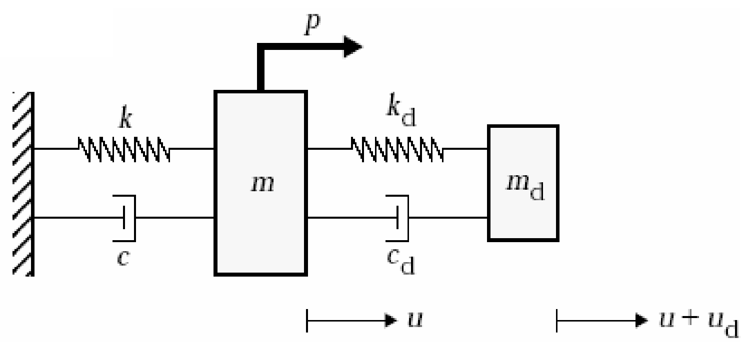


Figure 2: SDOF-TMD system [JJC: Figure 4.1]

– Undamped SDOF system with undamped TMD: $c = 0$ and $c_d = 0$

– Undamped SDOF system with damped TMD: $c = 0$ and $c_d \neq 0$

– Damped SDOF system with damped TMD: $c \neq 0$ and $c_d \neq 0$

(Q: Why there is no damped SDOF systems with undamped TMD?)

– **Undamped SDOF system with undamped TMD:** $c = 0$ **and** $c_d = 0$

∗ Governing equation:

$$m_d\left[\ddot{u}_d + \ddot{u}\right] + k_d u_d = -m_d a_g = -m_d \ddot{u}_g \tag{1}$$

$$m\ddot{u} + ku - k_d u_d = -m_d a_g + p = -m_d \ddot{u}_g + p \tag{2}$$

∗ Solution:

$$u(t) = \hat{u}\sin\omega_p t \tag{3}$$

$$u_d(t) = \hat{u}_d \sin\omega_p t \tag{4}$$

where

$$a_g(t) = \ddot{u}_g(t) = \hat{a}_g \sin\omega_p t \tag{5}$$

$$p(t) = \hat{p} \sin \omega_p t \tag{6}$$

$$\boxed{\hat{u} = \frac{\hat{p}}{k}\left(\frac{1-\rho_d^2}{D_1}\right) - \frac{m\hat{a}_g}{k}\left(\frac{1+\bar{m}-\rho_d^2}{D_1}\right)} \tag{7}$$

$$\boxed{\hat{u}_d = \frac{\hat{p}}{k_d}\left(\frac{\bar{m}\rho^2}{D_1}\right) - \frac{m\hat{a}_g}{k_d}\left(\frac{\bar{m}}{D_1}\right)} \tag{8}$$

$$\boxed{D_1 = [1-\rho^2]\,[1-\rho_d^2] - \bar{m}\rho^2} \tag{9}$$

$$\bar{m} = \frac{m_d}{m} \tag{10}$$

$$\rho = \frac{\omega_p}{\omega} = \frac{\omega_p}{\sqrt{\dfrac{k}{m}}} \tag{11}$$

$$\rho_d = \frac{\omega_p}{\omega_d} = \frac{\omega_p}{\sqrt{\dfrac{k_d}{m_d}}} \tag{12}$$

∗ For $1-\rho_d^2+\bar{m}=0$, $D_1 = 1$ and

$$\hat{u} = \frac{\hat{p}}{k} \tag{13}$$

$$\hat{u}_d = -\frac{\hat{p}}{k_d}\rho^2 + \frac{m\hat{a}_g}{k_d} \tag{14}$$

leading to the **optimal damper frequency**

$$\boxed{\omega_d|_{\text{opt}} = \frac{\omega_p}{\sqrt{1+\bar{m}}}} \tag{15}$$

which determines the **optimal damper stiffness**

$$\boxed{k_d|_{\text{opt}} = \omega_d|_{\text{opt}}^2 \cdot m_d = \frac{\omega_p^2 m\bar{m}}{1+\bar{m}}} \tag{16}$$

Finally, the **maximum damper displacement** at the optimal damper frequency is

$$\boxed{\hat{u}_d = \frac{1+\bar{m}}{\bar{m}}\left(\left|\frac{\hat{p}}{k}\right| + \left|\frac{\hat{a}_g}{\omega_p^2}\right|\right)} \tag{17}$$

∗ **Design steps:**

1. Use the condition $1-\rho_d^2+\bar{m}=0$.
2. Compute $\omega_d|_{\text{opt}}=\dfrac{\omega_p}{\sqrt{1+\bar{m}}}$.
3. Compute $k_d|_{\text{opt}}$.

– **Undamped SDOF system with damped TMD:** $c=0$ **and** $c_d\neq 0$

* Governing equation:

$$m_d\left[\ddot{u}_d+\ddot{u}\right]+k_du_d+c_d\dot{u}_d=-m_da_g=-m_d\ddot{u}_g \quad (18)$$
$$m\ddot{u}+ku-k_du_d-c_d\dot{u}_d=-m_da_g+p=-m_d\ddot{u}_g+p \quad (19)$$

* Solution:

$$u(t)=\bar{u}\exp\left(i\omega_pt\right) \quad (20)$$
$$u_d(t)=\hat{u}_d\exp\left(i\omega_pt\right) \quad (21)$$

where

$$a_g(t)=\hat{a}_g\exp\left(i\omega_pt\right) \quad (22)$$
$$p(t)=\hat{p}\exp\left(i\omega_pt\right) \quad (23)$$
$$\bar{u}=\frac{\hat{p}}{kD_2}\left[f^2-\rho^2+i2\xi_d\rho f\right]$$
$$-\frac{m\bar{a}_g}{kD_2}\left[\left(1+\bar{m}\right)f^2-\rho^2+i2\xi_d\rho f\left(1+\bar{m}\right)\right] \quad (24)$$
$$\Rightarrow \boxed{\bar{u}=\frac{\hat{p}}{k}H_1\exp\left(i\delta_1\right)-\frac{m\hat{a}_g}{k}H_2\exp\left(i\delta_2\right)} \quad (25)$$
$$\hat{u}_d=\frac{\hat{p}\rho^2}{kD_2}-\frac{m\hat{a}_g}{kD_2}$$
$$\Rightarrow \boxed{\bar{u}_d=\frac{\hat{p}}{k}H_3\exp\left(-i\delta_3\right)-\frac{m\hat{a}_g}{k}H_4\exp\left(-i\delta_3\right)} \quad (26)$$
$$\boxed{D_2=\left[1-\rho^2\right]\left[f^2-\rho^2\right]-\bar{m}\rho^2f^2+i2\xi_d\rho f\left[1-\rho^2\left(1+\bar{m}\right)\right]} \quad (27)$$
$$f=\frac{\omega_d}{\omega}=\frac{\rho}{\rho_d} \quad (28)$$

The H factors are

$$H_1=\frac{\sqrt{\left(f^2-\rho^2\right)^2+\left(2\xi_d\rho f\right)^2}}{|D_2|} \quad (29)$$

$$H_2 = \frac{\sqrt{\left[(1+\bar{m})\, f^2 - \rho^2\right]^2 + \left[2\xi_d \rho f\,(1+\bar{m})\right]^2}}{|D_2|} \tag{30}$$

$$H_3 = \frac{\rho^2}{|D_2|} \tag{31}$$

$$H_4 = \frac{1}{|D_2|} \tag{32}$$

$$|D_2| = \sqrt{\Re\left(D_2\right)^2 + \Im\left(D_2\right)^2} \tag{33}$$

$$\Re\left(D_2\right) = \left[1-\rho^2\right]\left[f^2-\rho^2\right] - \bar{m}\rho^2 f^2 \tag{34}$$

$$\Im\left(D_2\right) = 2\xi_d \rho f\left[1-\rho^2\left(1+\bar{m}\right)\right] \tag{35}$$

with

$$\delta_1 = \alpha_1 - \delta_3 \tag{36}$$

$$\delta_2 = \alpha_2 - \delta_3 \tag{37}$$

$$\delta_3 = \tan^{-1}\frac{\Im\left(D_2\right)}{\Re\left(D_2\right)} \tag{38}$$

$$\alpha_1 = \tan^{-1}\frac{2\xi_d \rho f}{f^2-\rho^2} \tag{39}$$

$$\alpha_2 = \tan^{-1}\frac{2\xi_d \rho f\left(1+\bar{m}\right)}{\left(1+\bar{m}\right) f^2-\rho^2} \tag{40}$$

Usually, $\bar{m} \in [0.01, 0.1]$. In the case of $\bar{m} \leq 0.05$, $H_1 \approx H_2$ and $\delta_1 \approx \delta_2$.

* Consider the optimization of TMD for ground motion (related to H_2) with $\bar{m} = 0.01$ and $f = 1$. Figure 3 shows the interaction between H_2 and ρ for various ρ_d. The optimal solution of ρ_d or ξ_d results in the minimum value of H_2, which is determined by

$$\left|1-\rho_1^2\left(1+\bar{m}\right)\right| = \left|1-\rho_2^2\left(1+\bar{m}\right)\right| \tag{41}$$

leading to the **optimal frequency ratio**, f_{opt}, and the optimal damper frequency, $\omega_d|_{\text{opt}}$.

$$\boxed{f_{\text{opt}} = \frac{\sqrt{1-0.5\bar{m}}}{1+\bar{m}}} \tag{42}$$

$$\boxed{\omega_d|_{\text{opt}} = f_{\text{opt}}\,\omega} \tag{43}$$

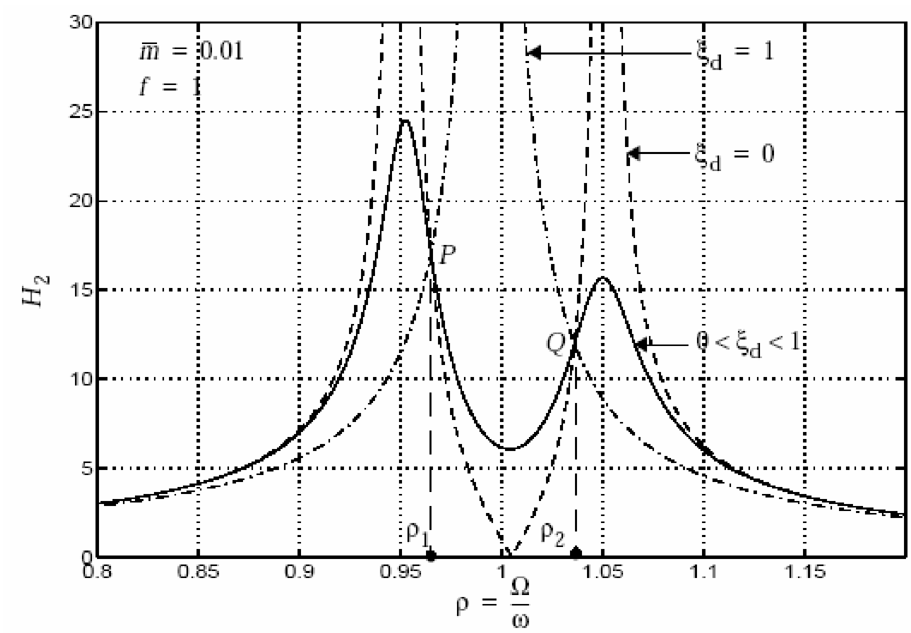


Figure 3: Plot of H_2 versus ρ [JJC: Figure 4.15]

which determines the optimal damper stiffness, $k_d|_{\text{opt}}$.

$$\boxed{k_d|_{\text{opt}} = \omega_d|_{\text{opt}}^2 \cdot m_d = \frac{1 - 0.5\bar{m}}{(1 + \bar{m})^2} \cdot \frac{k}{m} \cdot m_d} \tag{44}$$

The optimal value of H_2 is also found.

$$\boxed{H_2|_{\text{opt}} = \frac{1 + \bar{m}}{\sqrt{0.5\bar{m}}}} \tag{45}$$

which is associated with two optimal loading frequency ratios, $\rho_{1,2}|_{\text{opt}}$.

$$\boxed{\rho_{1,2}|_{\text{opt}} = \sqrt{\frac{1 \pm \sqrt{0.5\bar{m}}}{1 + \bar{m}}}} \tag{46}$$

The result is shown in Figure 4. Also, the optimal damping ratio for the TMD at the optimal tuning frequency is

$$\boxed{\xi_d|_{\text{opt}} = \sqrt{\frac{\left(3 - \sqrt{0.5\bar{m}}\right)\bar{m}}{8\,(1 + \bar{m})\,(1 - 0.5\bar{m})}}} \tag{47}$$

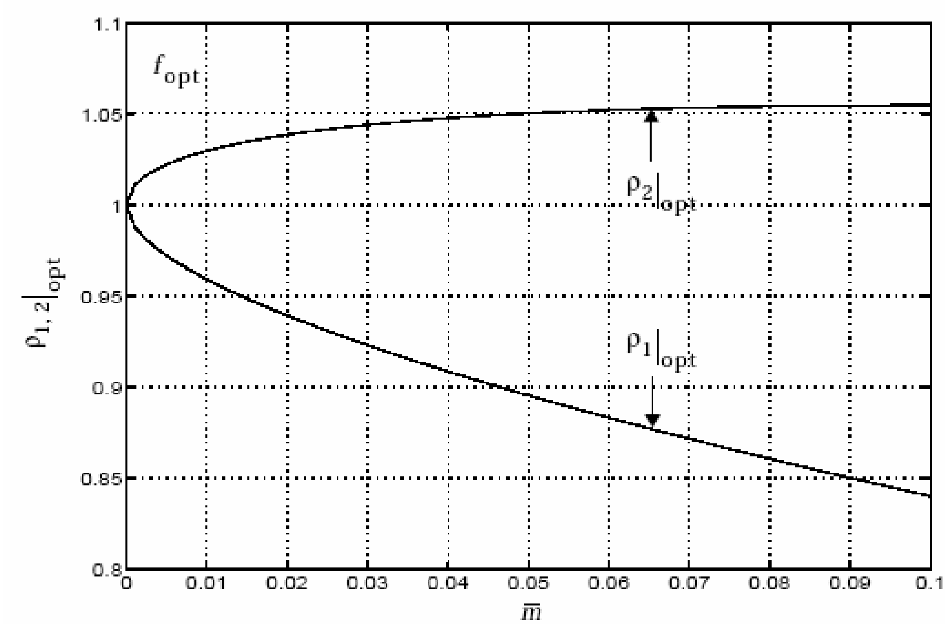


Figure 4: Plot of $\rho_{1,2}|_{\text{opt}}$ versus $\bar{m}$ [JJC: Figure 4.18]

The relationship between $H_2|_{\text{opt}}$ and $\bar{m} = \dfrac{m_d}{m}$ is provided in Figure 5. With same optimal parameters, $H_4|_{\text{opt}}$ versus $\bar{m}$ is shown in Figure 6. Note that H_2 is associated with the dynamic response of primary mass (system) and H_4 is associated with the TMD. The ratio of maximum TMD amplitude to maximum system amplitude, $\dfrac{H_4}{H_2}$, is provided in Figure Notice that the optimal parameters are derived from H_2 rather than H_4. This may result in uneven values of two peaks in H_4, as shown in a numerical example in Figure 8.

– **Design steps:**

1. Determine $H_2|_{\text{opt}}$ and $H_4|_{\text{opt}}$.
2. Determine $\bar{m}$.
3. Determine f_{opt}.
4. Compute ω_d.
5. Compute k_d.
6. Determine $\xi_d|_{\text{opt}}$.
7. Compute c_d.

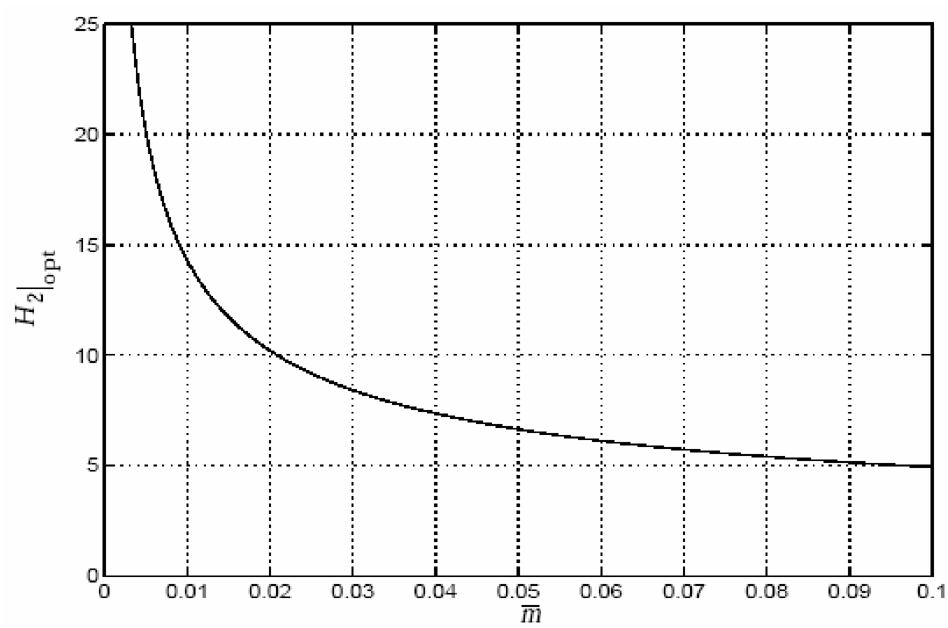


Figure 5: Plot of $H_2|_{\text{opt}}$ versus $\bar{m}$ [JJC: Figure 4.20]

- **Damped SDOF system with damped TMD:** $c \neq 0$ **and** $c_d \neq 0$

  – Governing equation:

$$m_d\left[\ddot{u}_d + \ddot{u}\right] + k_d u_d + c_d \dot{u}_d = -m_d a_g = -m_d \ddot{u}_g \tag{48}$$

$$m\ddot{u} + ku - k_d u_d - c_d \dot{u}_d + c\dot{u} = -m_d a_g + p = -m_d \ddot{u}_g + p \tag{49}$$

  – Solution:

$$\boxed{\bar{u} = \frac{\hat{p}}{k} H_5 \exp\left(i\delta_5\right) - \frac{m\hat{a}_g}{k} H_6 \exp\left(i\delta_6\right)} \tag{50}$$

$$\boxed{\hat{u}_d = \frac{\hat{p}}{k} H_7 \exp\left(-i\delta_7\right) - \frac{m\hat{a}_g}{k} H_8 \exp\left(-i\delta_8\right)} \tag{51}$$

$$\tag{52}$$

  where

$$H_5 = \frac{\sqrt{\left(f^2 - \rho^2\right)^2 + \left(2\xi_d \rho f\right)^2}}{|D_3|} \tag{53}$$

$$H_6 = \frac{\sqrt{\left[\left(1+\bar{m}\right) f^2 - \rho^2\right]^2 + \left[2\xi_d \rho f\left(1+\bar{m}\right)\right]^2}}{|D_3|} \tag{54}$$

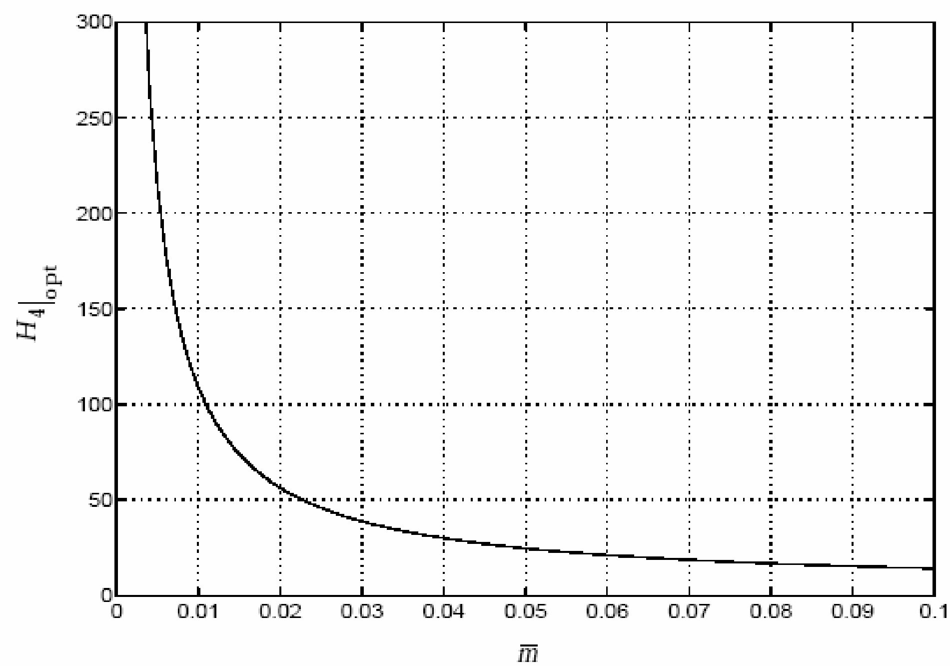


Figure 6: Plot of $H_4|_{\text{opt}}$ versus $\bar{m}$ [JJC: Figure 4.21]

$$H_7 = \frac{\rho^2}{|D_3|} \tag{55}$$

$$H_8 = \frac{\sqrt{1+(2\xi\rho)^2}}{|D_3|} \tag{56}$$

$$|D_3| = \left[\left(1-\rho^2\right)\left(f^2-\rho^2\right) - \bar{m}f^2\rho^2 - 4\xi\xi_d f\rho^2\right]^2 + 4\left\{\xi\rho\left(f^2-\rho^2\right) + \xi_d f\rho\left[1-\rho^2(1+\bar{m})\right]^2\right\} \tag{57}$$

$$\delta_5 = \alpha_1 - \delta_7 \tag{58}$$

$$\delta_6 = \alpha_2 - \delta_7 \tag{59}$$

$$\delta_7 = \tan^{-1}\left\{2\cdot\frac{\xi\rho\,(f^2-\rho^2) + \xi_d f\rho\,[1-\rho^2(1+\bar{m})]}{(1-\rho^2)(f^2-\rho^2) - \bar{m}f^2\rho^2 - 4\xi\xi_d f\rho^2}\right\} \tag{60}$$

$$\alpha_3 = \tan^{-1} 2\xi\rho \tag{61}$$

Since $|D_3|$ depends on ξ, f_{opt} and $\xi_d|_{\text{opt}}$ cannot be analytically determined.

– Design steps:
  1. Specify $\bar{m}$ and ξ for a range of f and ξ_d in H_5 versus ρ plots.
  2. Determine $\min[H_5]$ for a particular combination of f and ξ_d.

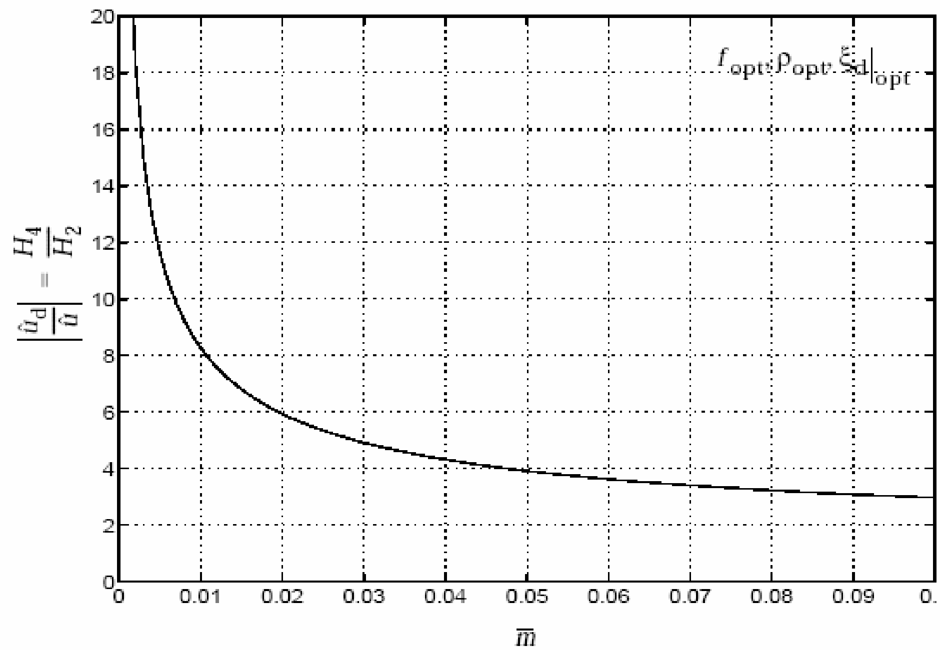


Figure 7: $\dfrac{H_4}{H_2}$ versus $\bar{m}$ [JJC: Figure 4.22]

3. Use different values of $\bar{m}$ and ξ to establish the interaction among these parameters. f_{opt}. Determine $\bar{m}$.
4. Determine f_{opt}.
5. Compute ω_d.
6. Compute k_d.
7. Determine $\xi_d|_{\text{opt}}$.
8. Compute c_d.

– Note that adding damping to the primary mass has an appreciable effect for small $\bar{m}$. When ξ is small, ratio $\dfrac{H_7}{H_5}$ is essentially independent of ξ.

- **Natural frequencies of some TMD systems**
  – Simple pendulum TMD: $\omega = \dfrac{g}{L}$
  – Liquid TMD: $\omega = \dfrac{2\pi}{L}\sqrt{gh}$
  – U-tube liquid TMD: $\omega = \sqrt{\dfrac{2g}{L}}$

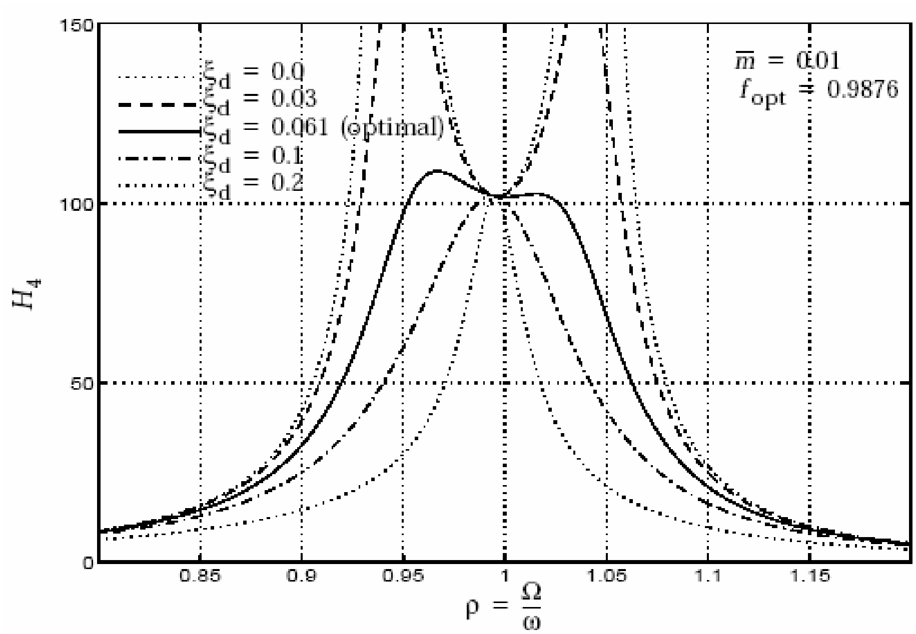


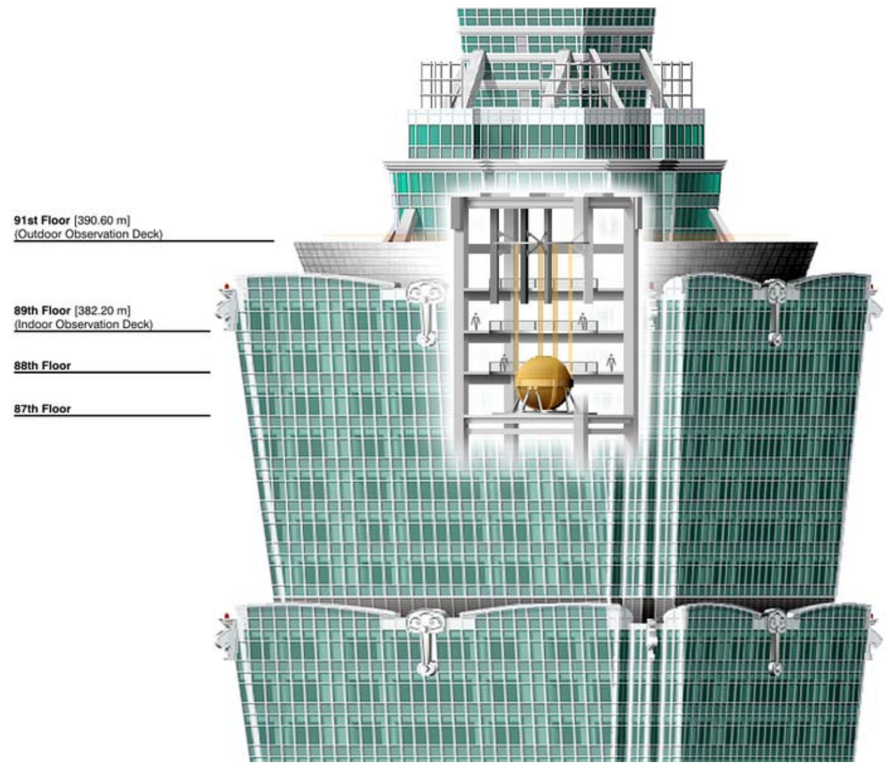
Figure 8: Plot of H_4 versus ρ for $\bar{m} = 0.01$, $f_{\text{opt}} = 0.9876$, and various ξ_d values including ξ_{opt} [JJC: Figure 4.24]

- **Two Examples of TMD systems**
  - The Landmark Tower, Yokohama, Japan
  - Taipei 101, Taipei, Taiwan

Reference

E. Matta and A. De Stefano (2009), "Robust design of mass-uncertain rolling-pendulum TMDs for the seismic protection of buildings", *Mechanical Systems and Signal Processing*, 23: 127-147.

Figure 9: Compound pendulum TMD system (340 tons) in the Landmark Tower, Yokohama, Japan



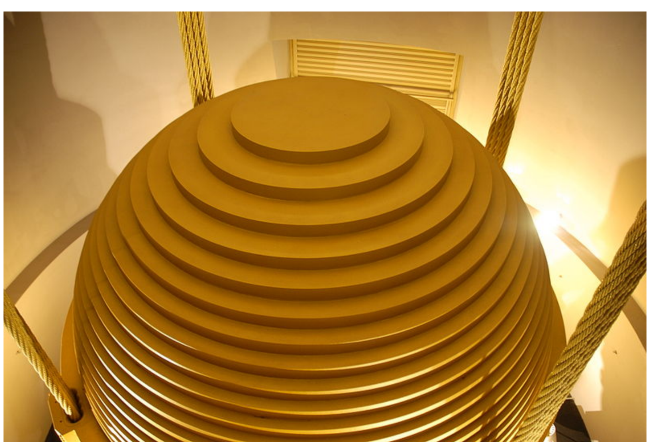


Figure 10: Simple pendulum TMD system (730 tons) in Taipei 101, Taipei, Taiwan