CIVE.5120 Structural Stability (3-0-3)
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Buckling of Thin Plates

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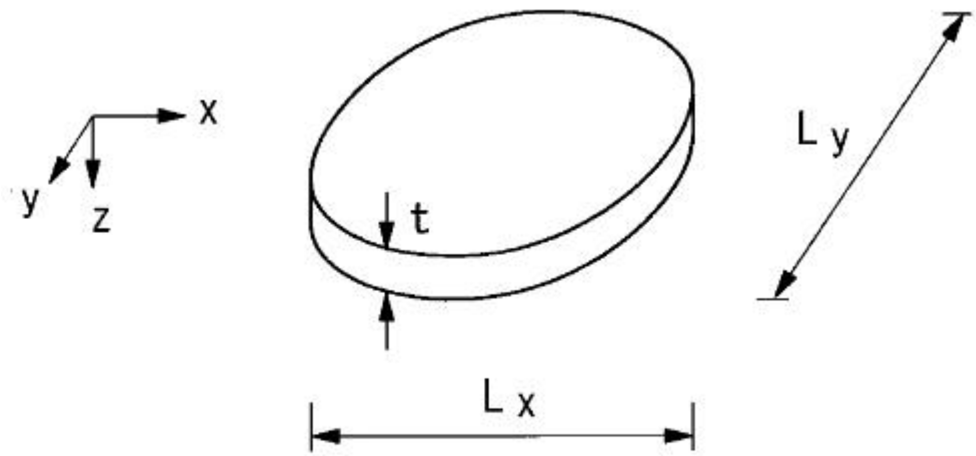
Outline

- Basic theory of thin plates
- Circular thin plates
- Thin plates in bending
- Buckling of uniformly loaded simply supported thin plates
- Buckling deformation of a thin tube of square cross section
- Buckling of rectangular thin plates under the action of shearing stresses
- Buckling of uniformly loaded circular thin plates
- Summary
- References

Thin Plates

- **Basic theory of thin plates**
  - **Assumptions:**
    - **One dimension (thickness) is much smaller than the other two dimensions (width and length) of the plate. →** $t << L_x, L_y$
    - **Shear stress is small; shear strains are small. →** $\sigma_z = 0; \varepsilon_z = \varepsilon_{xz} = \varepsilon_{yz} = 0$



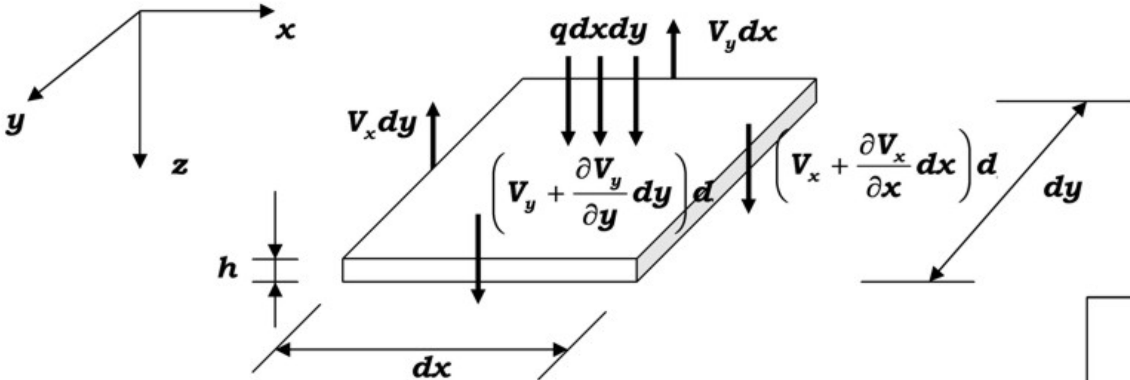
→ Thin plates must be thin enough to have small shear deformations but thick enough to accommodate in-plane/membrane forces.

Thin Plates

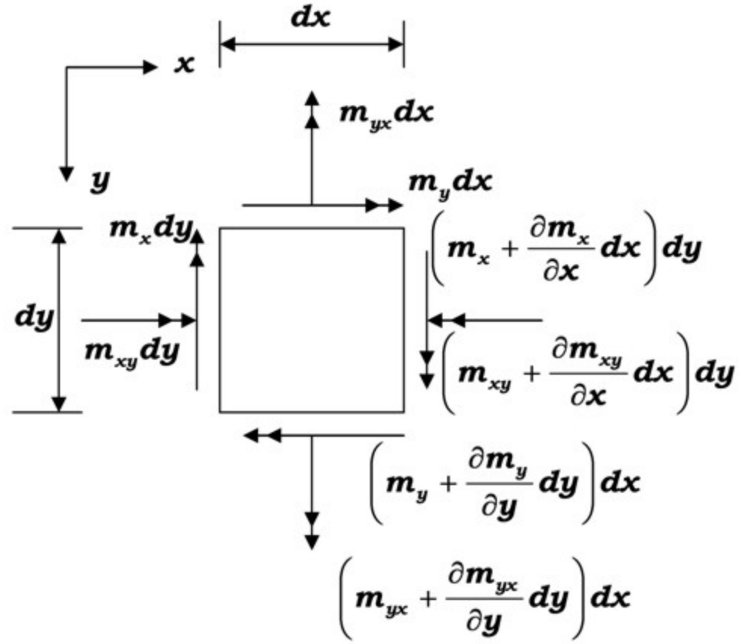
- **Basic theory of thin plates**
  - **Analysis approaches:**
    - **Elastic theory**
      - **Lagrange's 4th-order D.E. approach**
    - **Elastoplastic theory**
      - **Finite element analysis**
      - **Finite difference analysis**
    - **Approximate/empirical methods**
      - **Direct design method**
      - **Equivalent frame method**
      - **Moment distribution method**
    - **Limit analysis**
      - **Yield line theory (lower & upper bound analysis)**

Thin Plates

- **Basic theory of thin plates**
  - **Shear and shear stresses:**

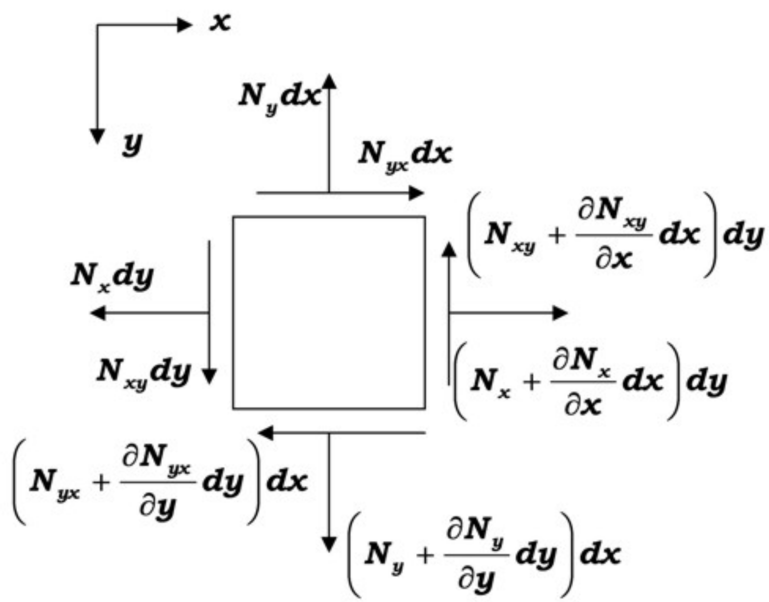


  - **Bending moments:**



Thin Plates

- **Basic theory of thin plates**
  - **Membrane forces:**



Thin Plates

- **Basic theory of thin plates**
  - **Strain-displacement relationships:**

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} \\
\varepsilon_y &= \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2} \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - z\frac{\partial^2 w_0}{\partial x \partial y}
\end{aligned}
$$

Thin Plates

- **Basic theory of thin plates**
  - **Equilibrium equations:**
    - **Force**

$$
\begin{aligned}
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + q_x &= 0 \\
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + q_y &= 0 \\
\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q_z &= 0
\end{aligned}
$$

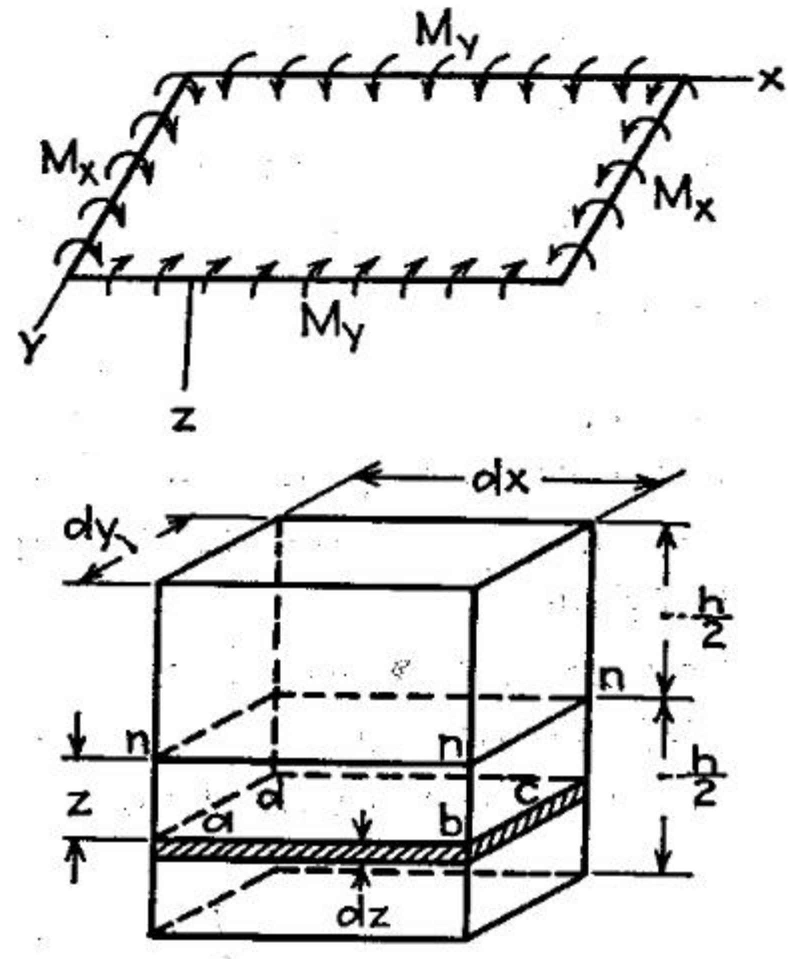
    - **Moment**

$$
\begin{aligned}
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - V_x &= 0 \\
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - V_y &= 0
\end{aligned}
$$

Thin Plates

- **Basic theory of thin plates**
  - **Resultant forces and moments:**

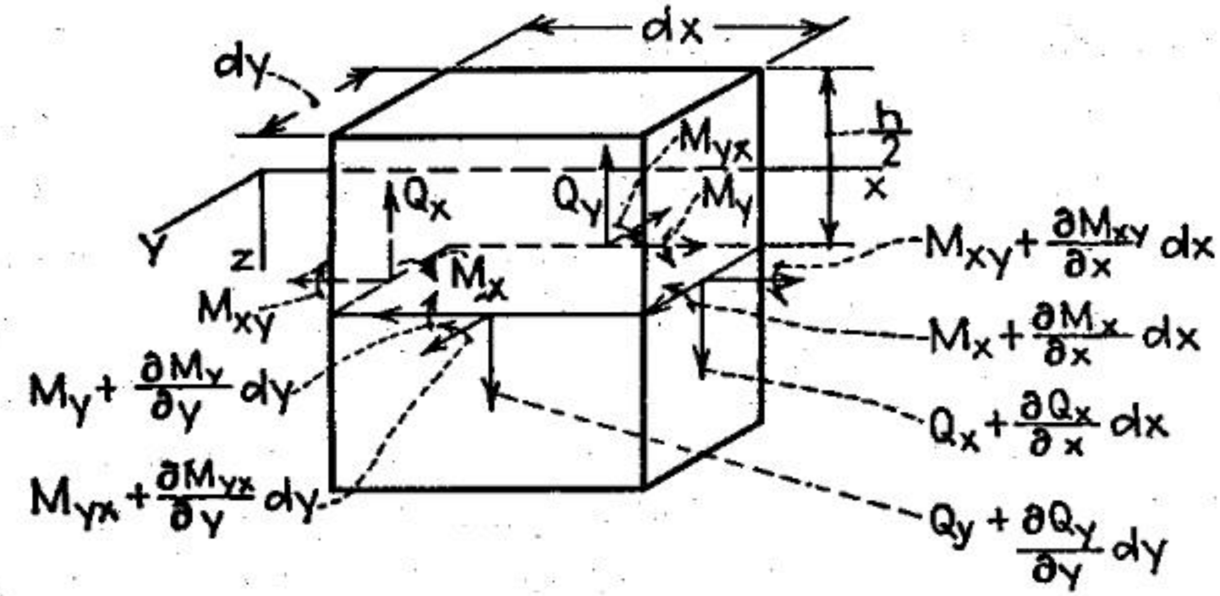
$$
\begin{aligned}
N_x &= \int_z \sigma_x dz \\
N_y &= \int_z \sigma_y dz \\
N_{xy} &= N_{yx} = \int_z \tau_{xy} dz \\
V_x &= \int_z \tau_{xz} dz \\
V_y &= \int_z \tau_{yz} dz \\
M_x &= \int_z \sigma_x z dz \\
M_y &= \int_z \sigma_y z dz \\
M_{xy} &= M_{yx} = \int_z \tau_{xy} z dz
\end{aligned}
$$



(Source: Timoshenko and Gere 1961)

Thin Plates

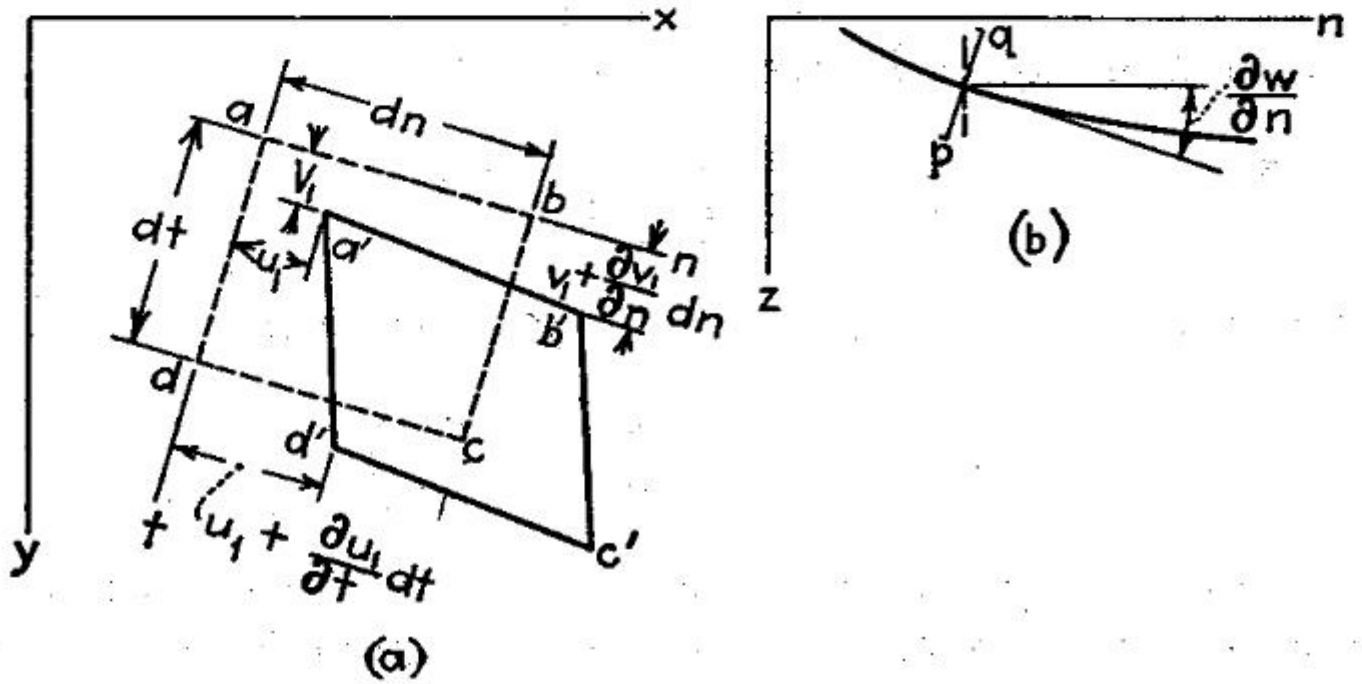
- **Basic theory of thin plates**
  - **Equilibrium in a thin plate cell:**



(Source: Timoshenko and Gere 1961)

Thin Plates

- **Basic theory of thin plates**
  - **Bending of a thin plate with small displacements:**



(Source: Timoshenko and Gere 1961)

Thin Plates

- **Basic theory of thin plates**
  - **Constitutive equations:**

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \Rightarrow$$

$$\begin{aligned} N_x &= \frac{Et}{1-\nu^2}\left(\frac{\partial u_0}{\partial x} + \nu \frac{\partial v_0}{\partial y}\right) \\ N_y &= \frac{Et}{1-\nu^2}\left(\frac{\partial v_0}{\partial y} + \nu \frac{\partial u_0}{\partial x}\right) \\ N_{xy} &= N_{yx} \frac{Et}{2(1+\nu)}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\right) \end{aligned}$$

$$\begin{aligned} M_x &= -D\left(\frac{\partial^2 w_0}{\partial x^2} + \nu \frac{\partial^2 w_0}{\partial y^2}\right) \\ M_y &= -D\left(\frac{\partial^2 w_0}{\partial y^2} + \nu \frac{\partial^2 w_0}{\partial x^2}\right) \\ M_{xy} &= M_{yx} = -(1-\nu) D \frac{\partial^2 w_0}{\partial x \partial y} \end{aligned}$$

Thin Plates

- **Basic theory of thin plates**
  - **Flexural rigidity of thin plates:**

$$D = \frac{Et^3}{12(1-\nu^2)}$$

  - **In-plane problems:**

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + q_x = 0$$
$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + q_y = 0$$

$$\begin{aligned} N_x &= \frac{Et}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right) \\ N_y &= \frac{Et}{1-\nu^2}\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right) \\ N_{xy} &= N_{yx} = \frac{Et}{2(1+\nu)}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \end{aligned}$$

Thin Plates

- **Basic theory of thin plates**
  - **Out-of-plane problems:**

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q_z = 0 \qquad M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - V_x = 0 \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2}\right)$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - V_y = 0 \qquad M_{xy} = M_{yx} = -(1-\nu)D\frac{\partial^2 w}{\partial x \partial y}$$

Thin Plates

- **Basic theory of thin plates**
  - **Out-of-plane problems: Governing equation**

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q_z = 0$$
$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - V_x = 0$$
$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - V_y = 0$$

$$\Rightarrow \quad \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q_z = 0$$

$$\Rightarrow \quad \boxed{D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = q_z}$$

$$\nabla^4 w = \frac{q_z}{D}$$

where

$$\begin{aligned} \nabla^4 &= \nabla^2 \nabla^2 \\ \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{aligned}$$

Thin Plates

- **Circular thin plates**
  - **General circular thin plates:**

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)w=\frac{q_z}{D}$$

$$\begin{aligned}
M_r &= -D\left[(1-\nu)\frac{\partial^2 w}{\partial r^2}+\nu\nabla^2 w\right]\\
M_\theta &= -D\left[\nabla^2 w+(1-\nu)\frac{\partial^2 w}{\partial r^2}\right]\\
M_{r\theta} &= M_{\theta r}=-D(1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\\
S_r &= V_r+\frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta}\\
S_\theta &= V_\theta+\frac{\partial M_{r\theta}}{\partial r}
\end{aligned}$$

Thin Plates

- **Circular thin plates**
  - **Axisymmetric circular thin plates:**

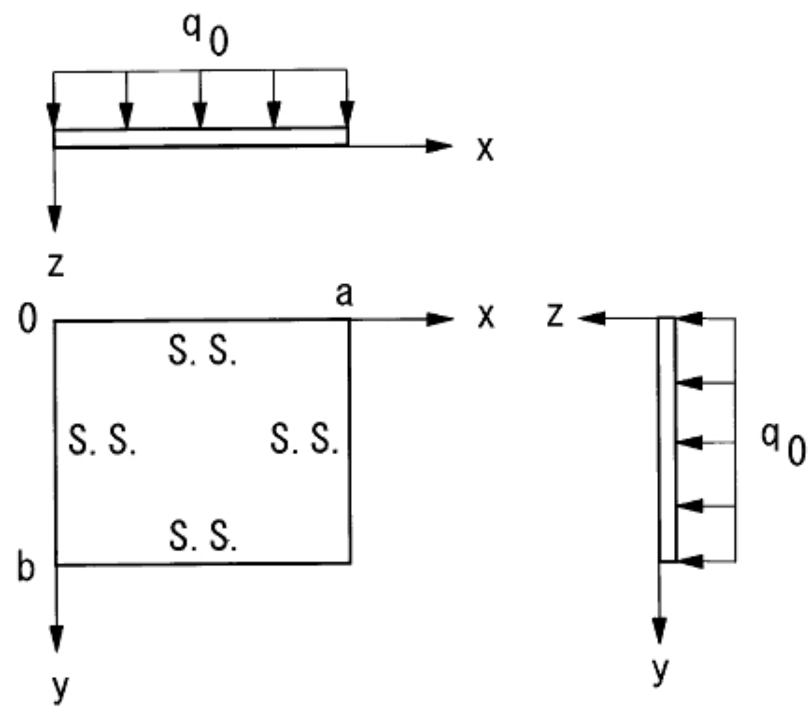
$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right] = \frac{q_z}{D}$$

$$\begin{aligned} M_r &= D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right) \\ M_\theta &= D\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right) \\ M_{r\theta} &= M_{\theta r} = 0 \end{aligned}$$

Thin Plates

- **Thin plates in bending**
  - Simply-supported rectangular plate subjected to uniform load

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q_0}{D}$$



<u>Boundary conditions:</u>

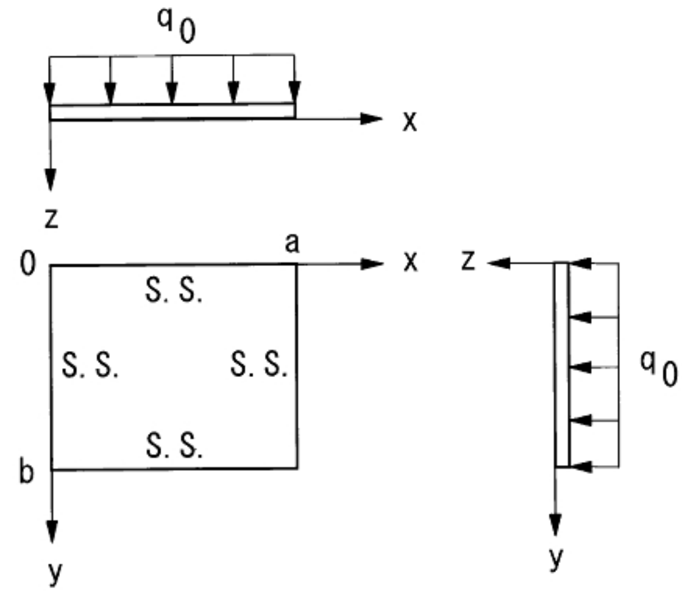
$$\begin{aligned} w &= 0, \quad M_x = 0 \quad \text{along} \quad x = 0, a \\ w &= 0, \quad M_y = 0 \quad \text{along} \quad y = 0, b \end{aligned}$$

$$\begin{aligned} w &= 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{along} \quad x = 0, a \\ w &= 0, \quad \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{along} \quad y = 0, b \end{aligned}$$

Thin Plates

- **Thin plates in bending**
  - Simply-supported rectangular plate subjected to uniform load

Solution:



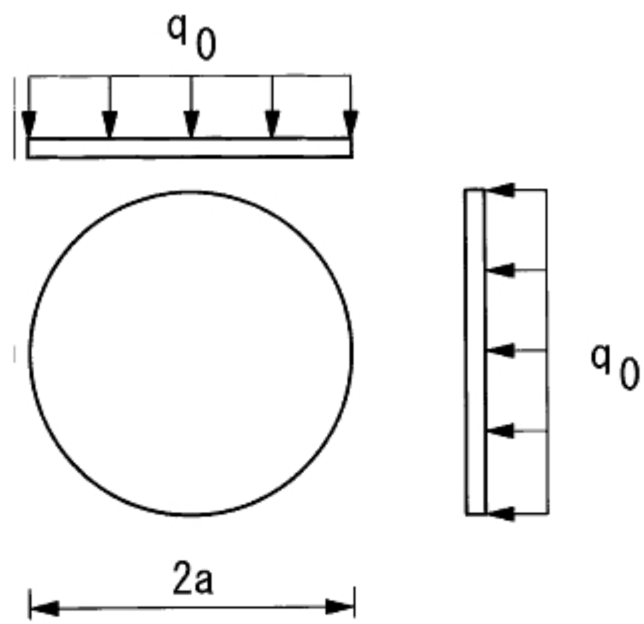
$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\boxed{w = \frac{16 q_0}{\pi^6 D} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}$$

Thin Plates

- **Thin plates in bending**
  - Axisymmetric circular plate with built-in edge subjected to uniform load

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right]=\frac{q_0}{D}$$



Boundary conditions:

$$w=\frac{dw}{dr}=0 \text{ at } r=a$$

Solutions:

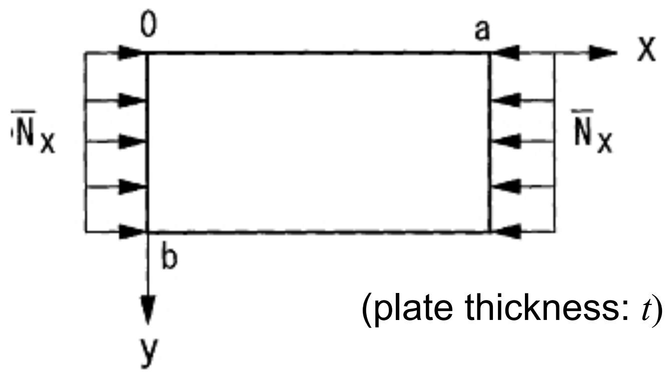
$$w=\frac{q_0 r^4}{64D}+A_1 r^2\ln r+A_2\ln r+A_3 r^2+A_4$$

$$\boxed{w=\frac{q_0 a^4}{64D}\left(\frac{r^2}{a^2}-1\right)^2}$$

Thin Plates

- **Buckling of uniformly loaded simply supported thin plates**

$$D\nabla^4 w + \overline{N}_x \frac{\partial^2 w}{\partial x^2} = 0$$



(plate thickness: t)

<u>Boundary conditions:</u>

$$w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{along } x = 0, a$$

$$w = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{along } y = 0, b$$

<u>Solution:</u>

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi x}{b}$$

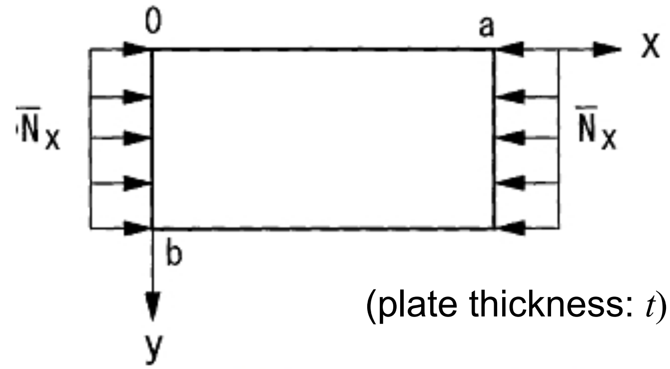
$$A_{mn} \left[\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 - \frac{\overline{N}_x}{D} \frac{m^2 \pi^2}{a^2} \right] = 0$$

Thin Plates

- **Buckling of uniformly loaded simply supported thin plates**

<u>Solution:</u>

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi x}{b}$$



(plate thickness: t)

$$\Rightarrow\quad A_{mn}\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 - \frac{\overline{N}_x}{D}\frac{m^2\pi^2}{a^2}\right] = 0$$

$$\Rightarrow\quad \overline{N}_x = \frac{\pi^2 D}{b^2}\left(m\frac{b}{a}+\frac{n^2}{m}\frac{a}{b}\right)^2$$

$$\Rightarrow\quad \boxed{\overline{N}_{xC} = \frac{k\pi^2 D}{b^2}}$$

where $\sigma_{xC} = \dfrac{\overline{N}_{xC}}{t} = k\dfrac{\pi^2 E}{12(1-\nu^2)}\dfrac{1}{(b/t)^2}$

$$k = \left(m\frac{b}{a}+\frac{1}{m}\frac{a}{b}\right)^2$$

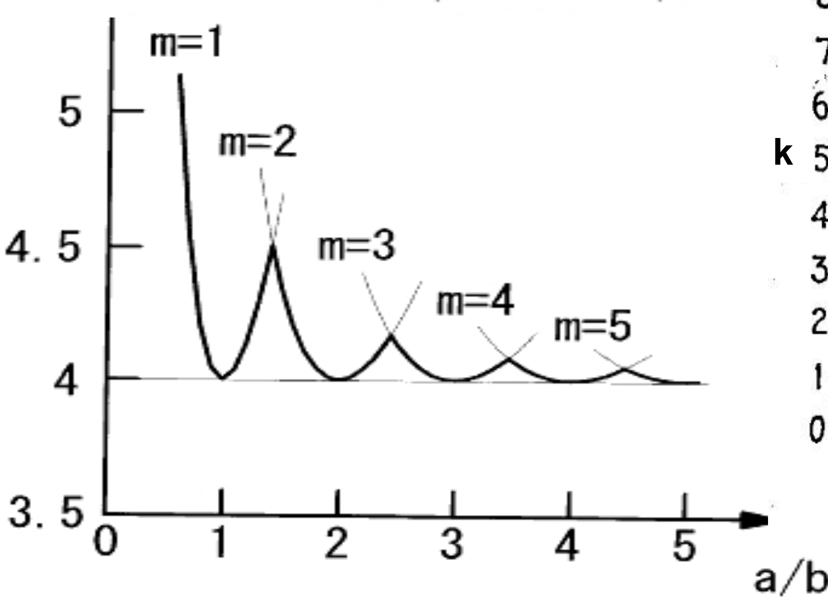
$$D = \frac{Et^3}{12(1-\nu^2)}$$

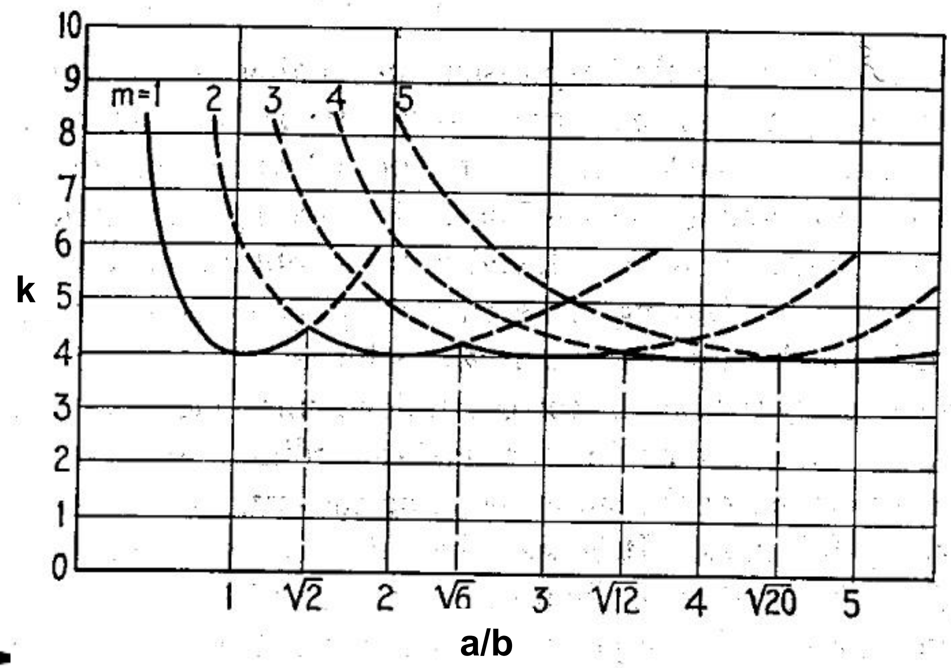
Thin Plates

- **Buckling of uniformly loaded simply supported thin plates**

$$\sigma_{xC} = \frac{\overline{N}_{xC}}{t} = k\frac{\pi^2 E}{12(1-\nu^2)}\frac{1}{(b/t)^2}$$

$$k = \left(m\frac{b}{a} + \frac{1}{m}\frac{a}{b}\right)^2$$





(Source: Timoshenko and Gere 1961)

Thin Plates

- **Buckling of uniformly loaded simply supported thin plates**
  - **Simplified solution:**

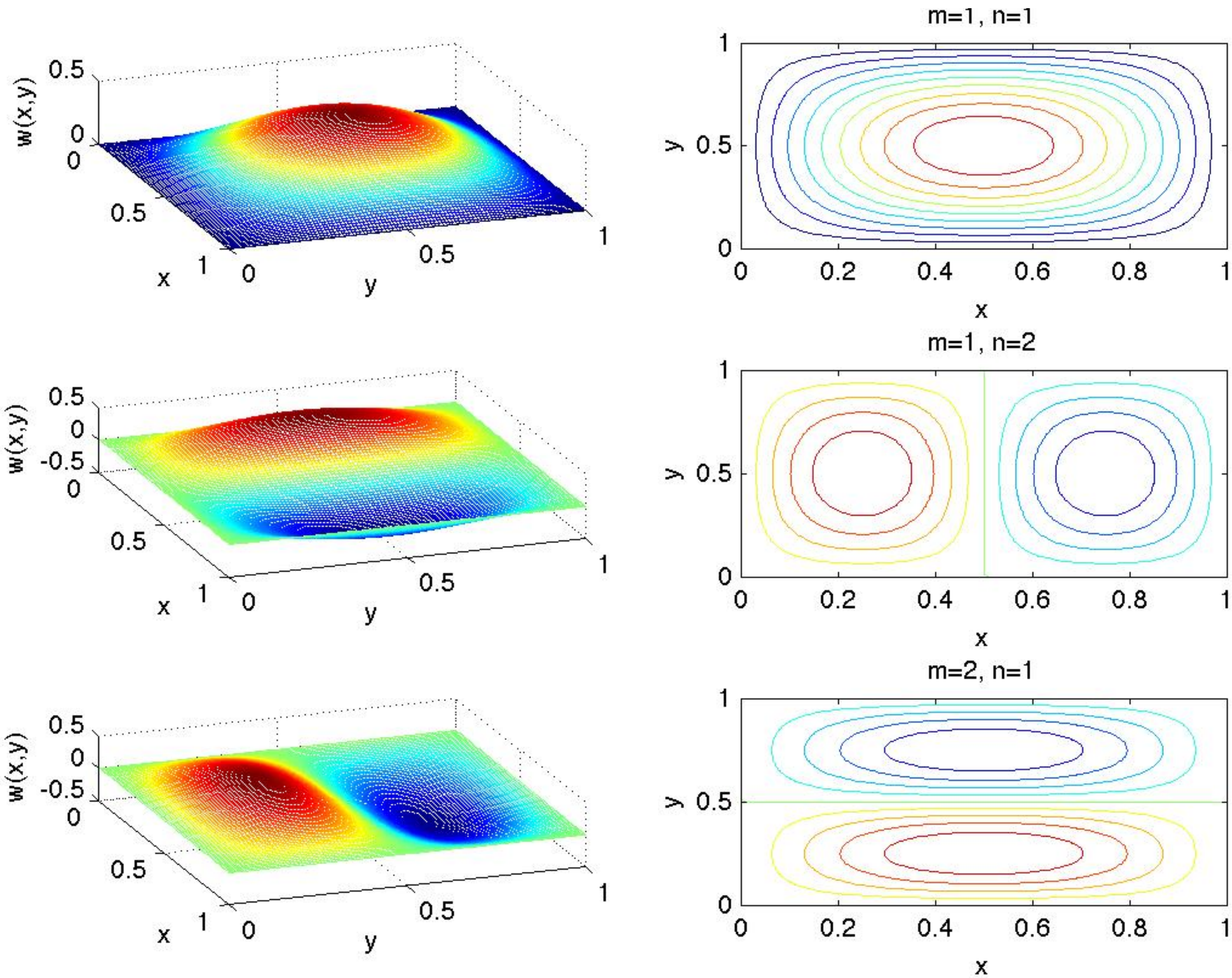
TABLE VALUES OF FACTOR k FOR UNIFORMLY COMPRESSED, SIMPLY SUPPORTED RECTANGULAR PLATES AND σ_{cr} IN PSI FOR $E = 30 \cdot 10^6$ PSI, $b/h = 100$, $\nu = 0.3$

a/b	0.2	0.3	0.4	0.5	0.6	0.7	0.8
k	27.0	13.2	8.41	6.25	5.14	4.53	4.20
σ_{cr}	73,200	35,800	22,800	16,900	13,900	12,300	11,400
a/b	0.9	1.0	1.1	1.2	1.3	1.4	1.41
k	4.04	4.00	4.04	4.13	4.28	4.47	4.49
σ_{cr}	11,000	10,800	11,000	11,200	11,600	12,100	12,200

(Source: Timoshenko and Gere 1961)

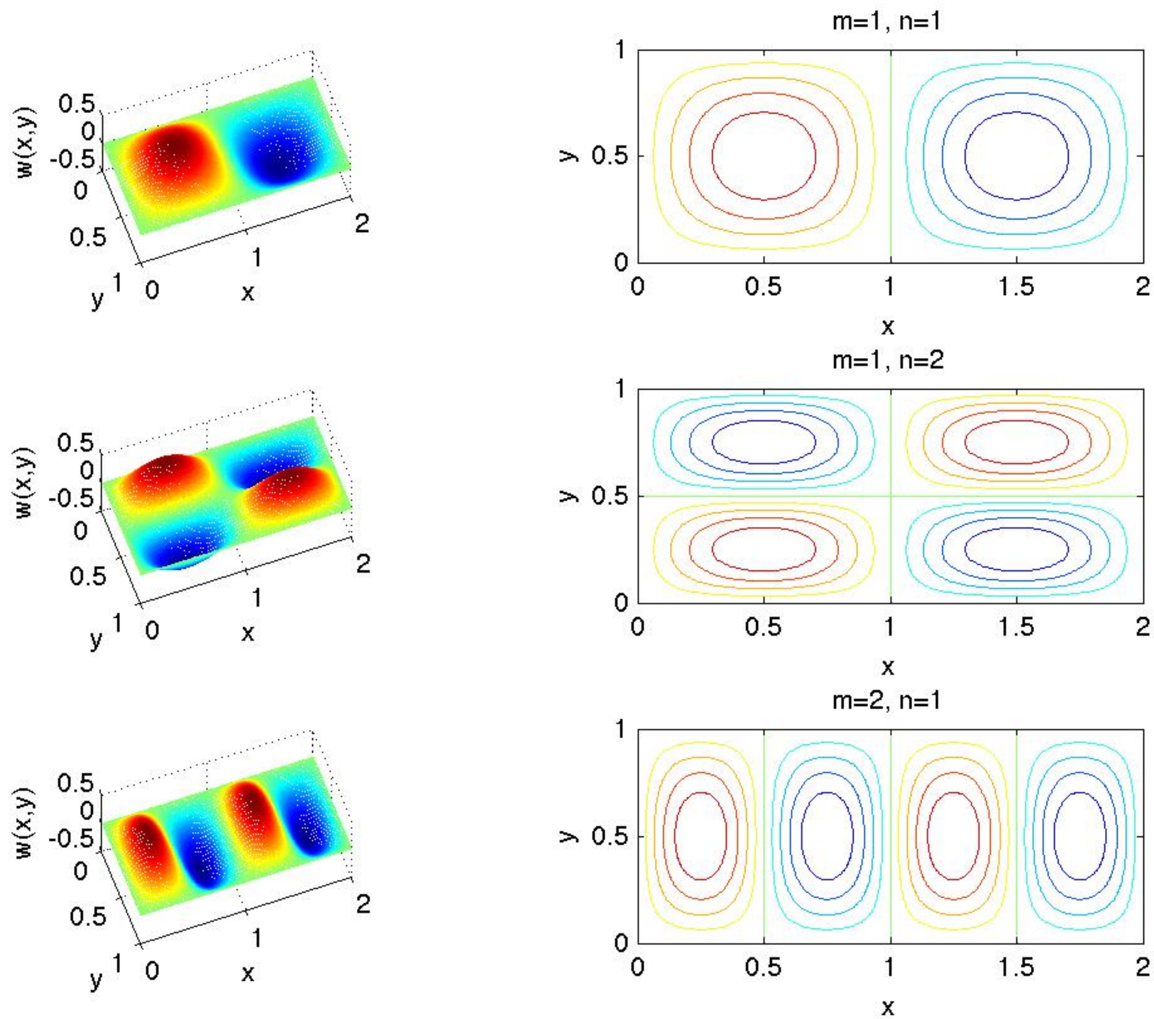
Buckling Modes of Thin Plates

- Buckling modes of a simply-supported thin plate
  - **a = 1, b = 1**
  - Mode (1, 1)
  - Mode (1, 2)
  - Mode (2, 1)



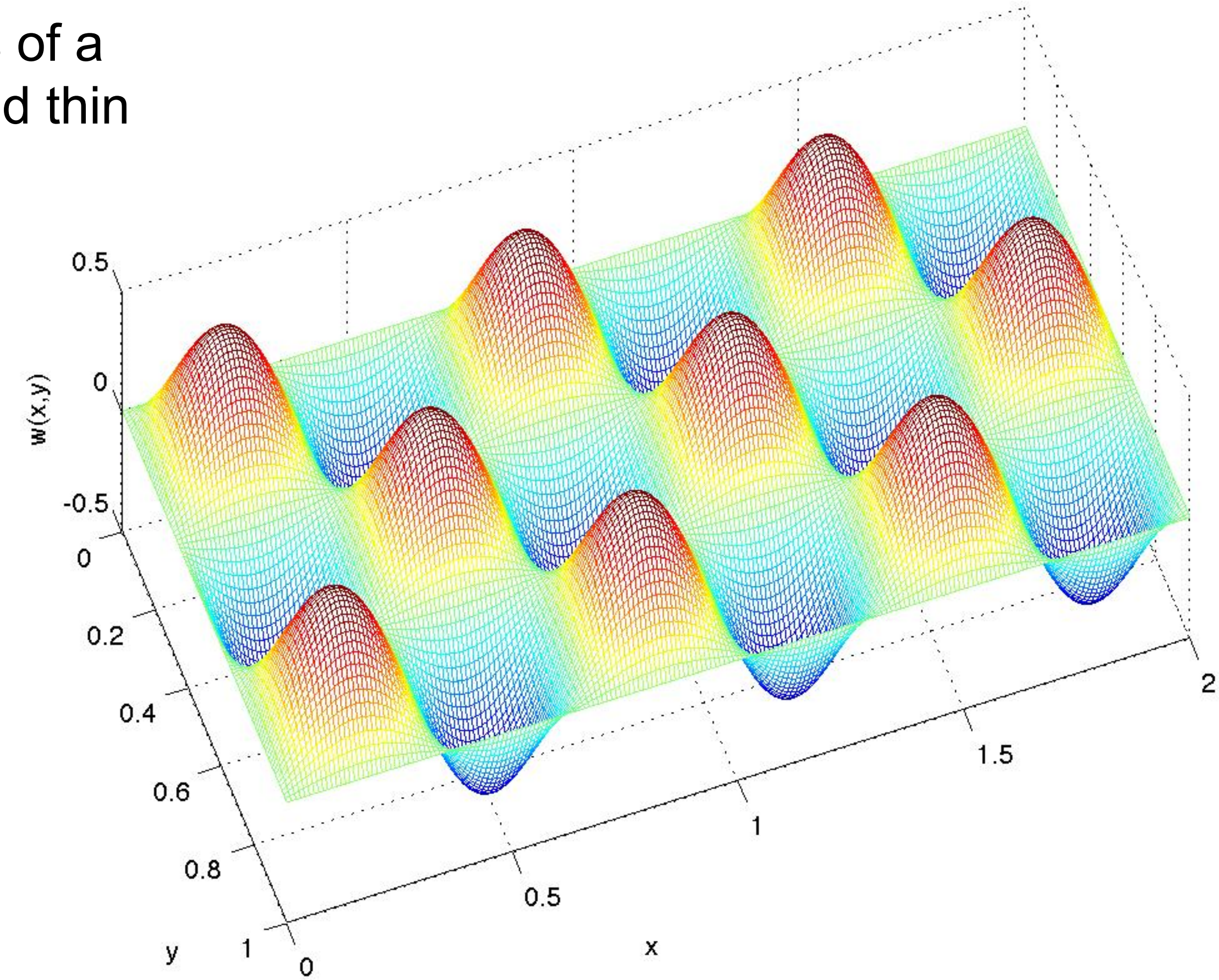
Buckling Modes of Thin Plates

- Buckling modes of a simply-supported thin plate
  - **a = 2, b = 1**
  - Mode (1, 1)
  - Mode (1, 2)
  - Mode (2, 1)



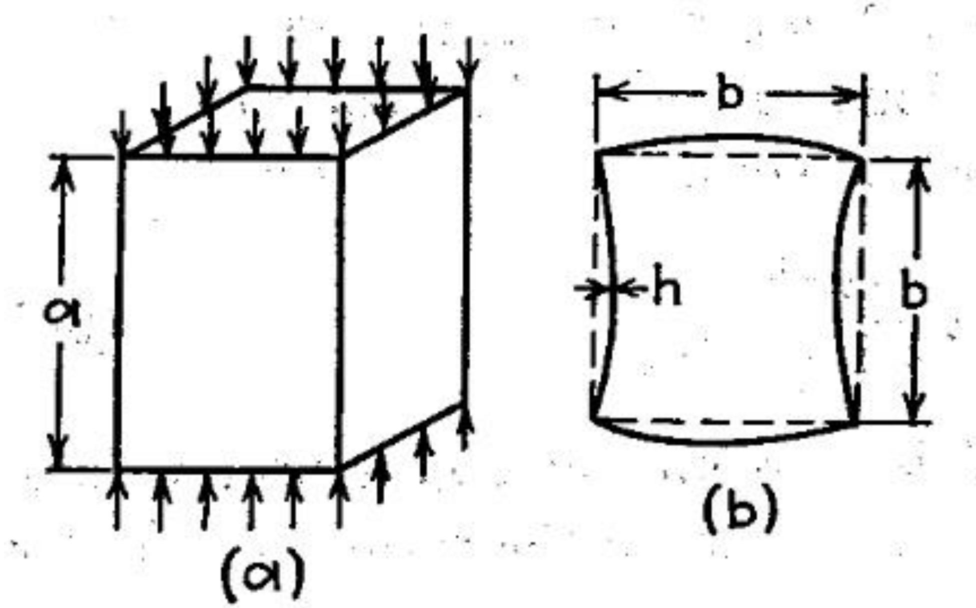
Buckling Modes

- Buckling modes of a simply-supported thin plate
  - **a = 2, b = 1**
  - Mode (3, 3)



Thin Plates

- **Buckling deformation of a thin tube of square cross section**

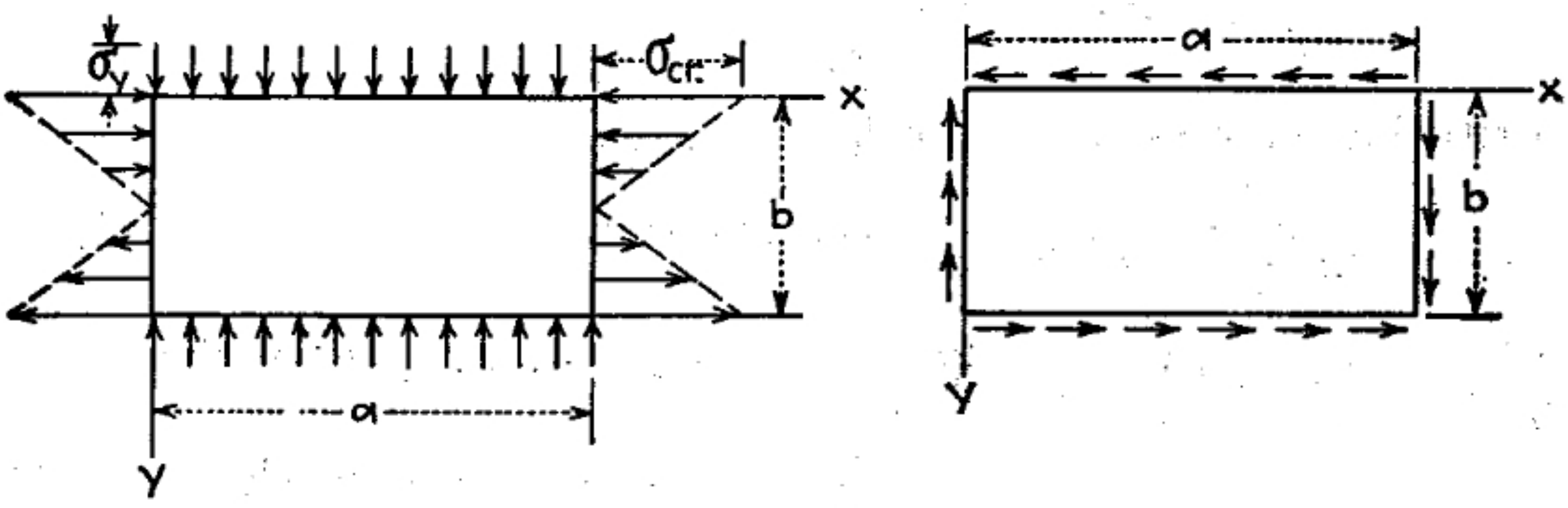


→ There is no bending moments acting between the sides of the buckled tube along the corners; each side is in the condition of a compressed rectangular plate with simply-supported edges.

(Source: Timoshenko and Gere 1961)

Thin Plates

- **Buckling of rectangular thin plates under the action of shearing stresses**



(Source: Timoshenko and Gere 1961)

Thin Plates

- **Buckling of rectangular thin plates under the action of shearing stresses**

Solution:

$$w = \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$N_{xy} = -\frac{abD}{32} \frac{\sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} a_{mn}{}^2 \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)^2}{\sum_{m} \sum_{n} \sum_{p} \sum_{q} a_{mn} a_{pq} \frac{mnpq}{(m^2 - p^2)(q^2 - n^2)}}$$

Thin Plates

- **Buckling of rectangular thin plates under the action of shearing stresses**

Simplified solution:

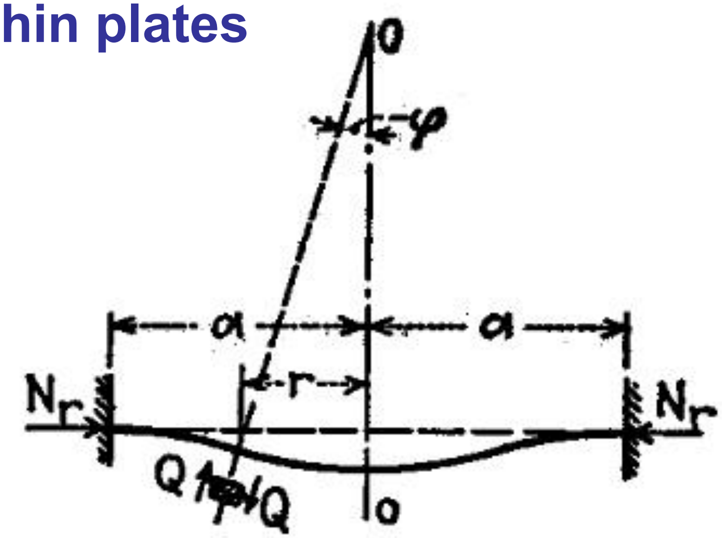
TABLE VALUES OF THE FACTOR k

a/b	1.0	1.2	1.4	1.5	1.6	1.8	2.0	2.5	3	4
k	9.34	8.0	7.3	7.1	7.0	6.8	6.6	6.1	5.9	5.7

(Source: Timoshenko and Gere 1961)

Thin Plates

- **Buckling of uniformly loaded circular thin plates**



Governing equation:

Boundary conditions:

Solution:

(Source: Timoshenko and Gere 1961)

Thin Plates

- **Review – Bessel functions**

First kind:

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\tfrac{1}{2}x\right)^{2m+\alpha}$$

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\mathrm{i}\,(n\tau - x\sin\tau)}\, \mathrm{d}\tau$$

$$J_{-n}(x) = (-1)^n J_n(x)$$

Second kind:

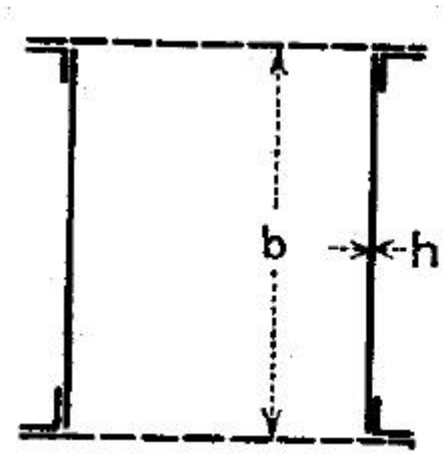
$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}.$$

$$Y_n(x) = \frac{1}{\pi} \int_0^{\pi} \sin(x\sin\theta - n\theta)\, d\theta - \frac{1}{\pi} \int_0^{\infty} \left[e^{nt} + (-1)^n e^{-nt}\right] e^{-x\sinh t}\, dt$$

$$Y_{-n}(x) = (-1)^n Y_n(x)$$

Thin Plates

- **Buckling analysis of a composite tube**



(a = b)

Summary

- Homogeneous, isotropic, elastic thin plates are considered.
- Buckling modes and shapes depend on plate geometry and the boundary condition (supports) of the plate.
- Thin plates are thin enough to permit small shear deformations but thick enough to permit membrane forces.
- Boundary conditions and the aspect ratio of thin plates are primarily responsible for the level of critical load of thin plates.

References

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