CIVE.5120 Structural Stability (3-0-3)
04/25/17

Buckling of Shells / Advanced Topics

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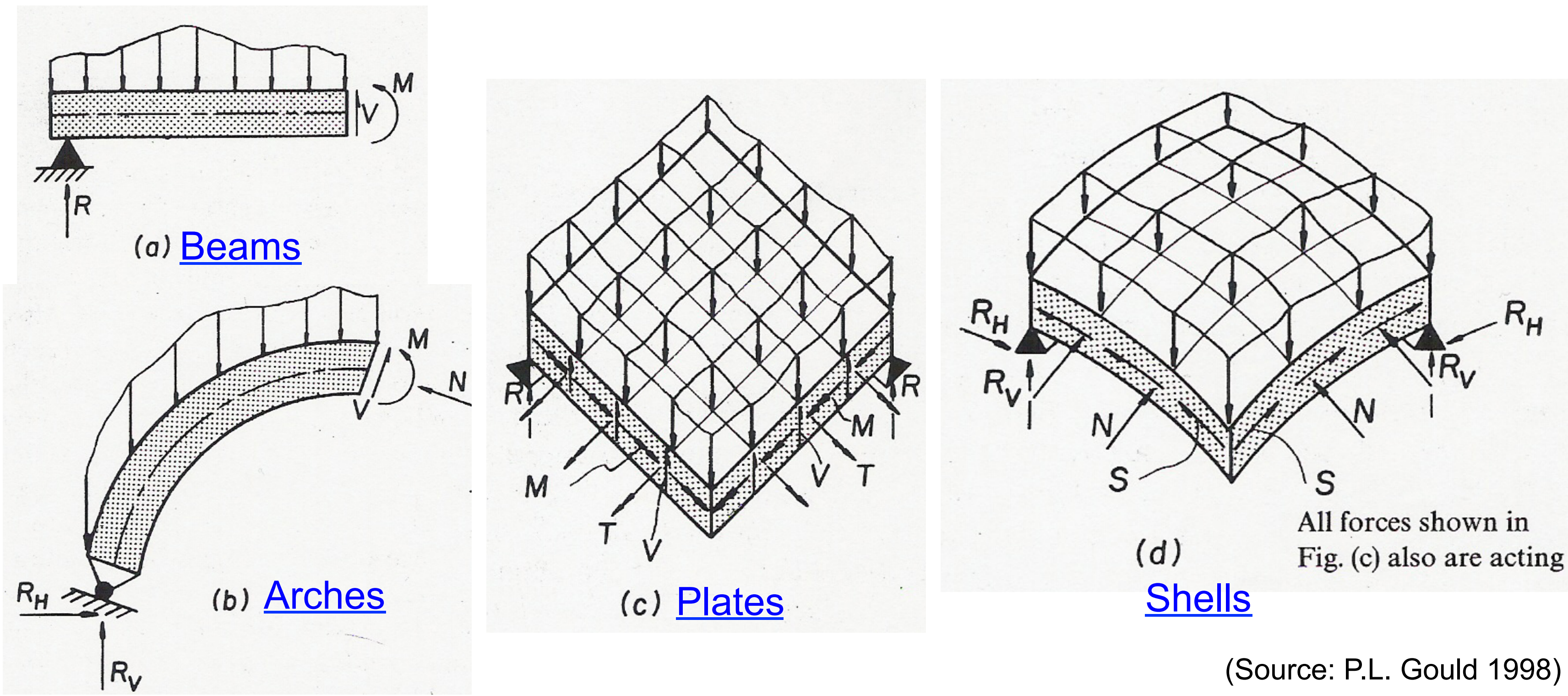
Outline

- Basic theory of shells
- Basic theory of thin shells
- Examples of shell structures
- Analysis of cylindrical shells
- Buckling of cylindrical shells
- Advanced Topics
  - The Routh-Hurwitz Theorem
  - The Lyapunov Theorems
    - The Lyapunov Stability Theorem
    - The Lyapunov Instability Theorem
  - Application of the Lyapunov Stability Theorems
    - Static and Dynamic Stability Problems
- Summary
- References

Thin Shells

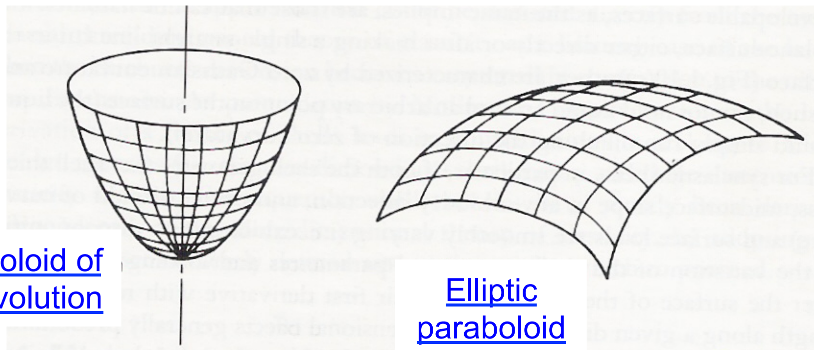
- **Basic theory of shells**
  - **Differences between plates and shells:**
    - **Shells carry membrane and bending forces → Shells are stronger than plates due to membrane forces.**

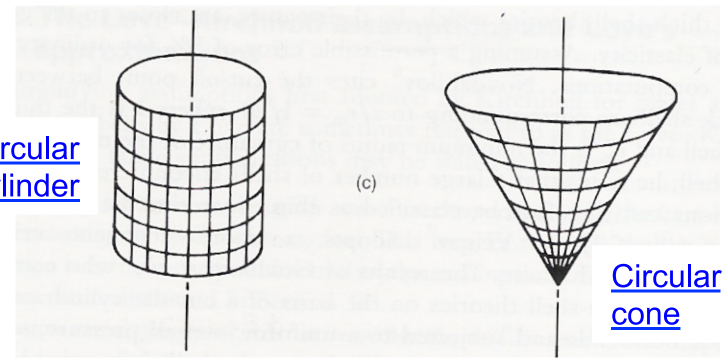


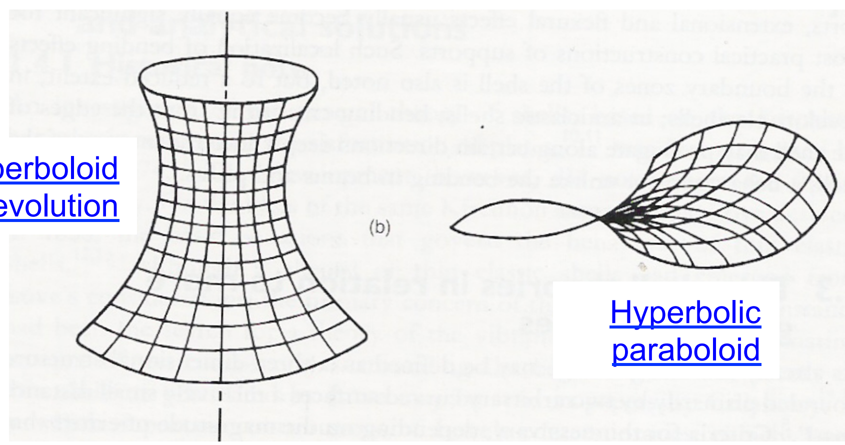
(Source: P.L. Gould 1998)

Shells

- **Basic theory of shells**
  - **Types of shells:**







(Source: P.L. Gould 1998)

Shells

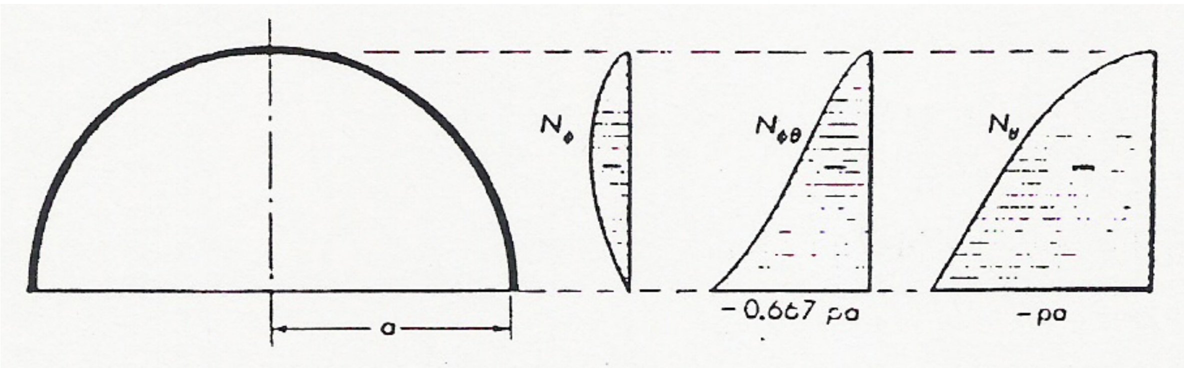
- **Basic theory of shells**
  - **The Flügge-Byrne Theory for Shells**
    - Strains and displacements that arise within the shells are small.
    - Second-order or higher-order approximations of shells
  - **The Mindlin-Reissner Theory for Thick Shells**
    - Straight lines that are normal to the mid-surface remains straight but not necessarily perpendicular to the mid-surface
    - Strains and displacements that arise within the shells are NOT small. → Shear strains are constant across the thickness of shells.
    - The direct stress acting in the direction normal to the shell middle surface is negligible.

Shells

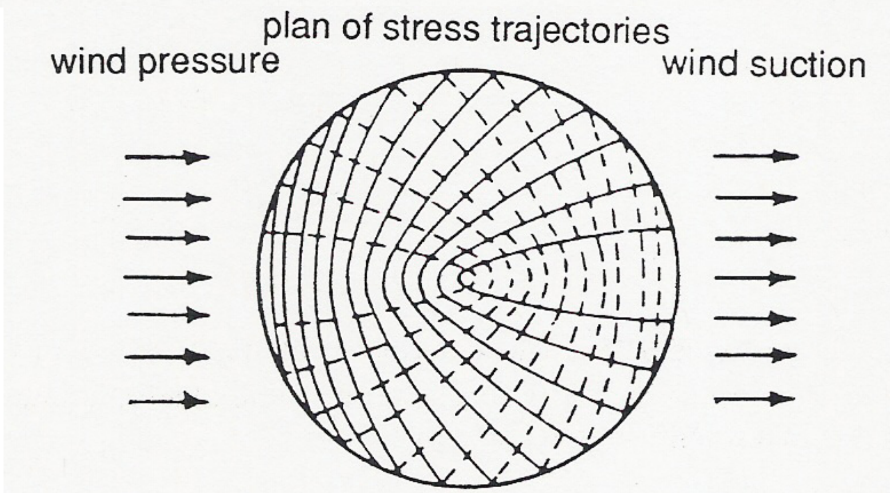
- **Basic theory of shells**
  - **Effect of shell thickness**
    - Very thick shells: 3D effects
    - Thick shells: stretching, bending and higher order transverse shear
    - Moderately thick shells: stretching, bending and first order transverse shear
    - Thin shells: stretching and bending energy considered but transverse shear neglected
    - Very thin shells: dominated by stretching effects. Also called membranes.
  - **Approaches of Analysis**
    - Energy method
    - Rayleigh-Ritz methods
    - Galerkin method

Shells

- **Basic theory of shells**
  - **Stress distribution in a spherical dome (paraboloid of revolution):**



Variations of internal membrane forces in a spherical dome subjected to lateral wind loading

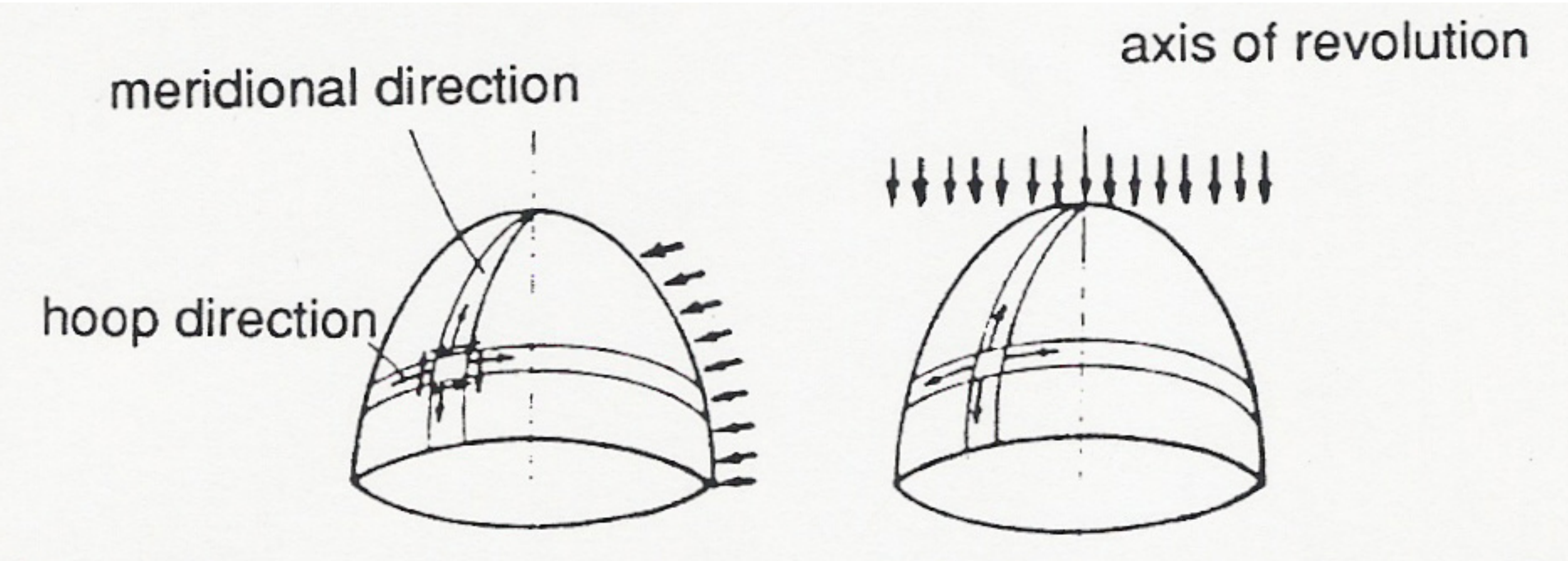


Stress trajectories in a spherical dome subjected to lateral wind loading

(Source: M. Farshad 1992)

Shells

- **Basic theory of shells**
  - **Effect of stress symmetricity in a parabolic dome:**

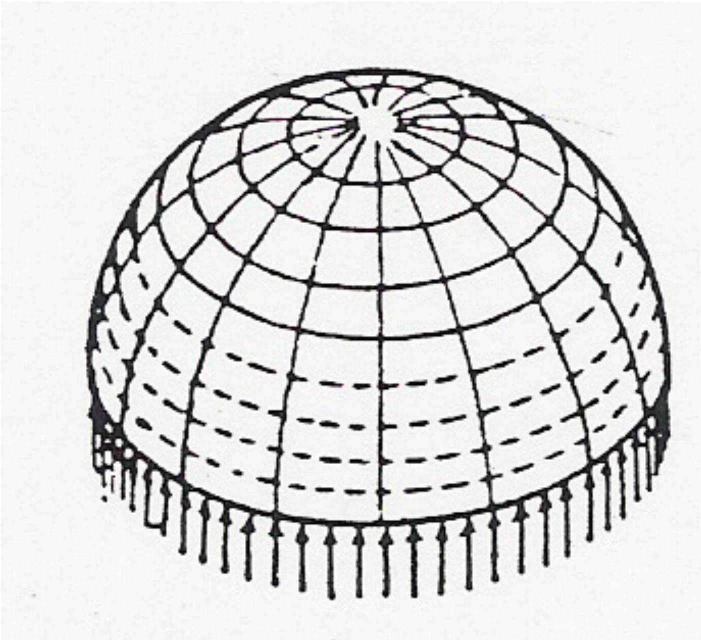


(a) Dome with non-axisymmetric load (b) Dome with axisymmetric load

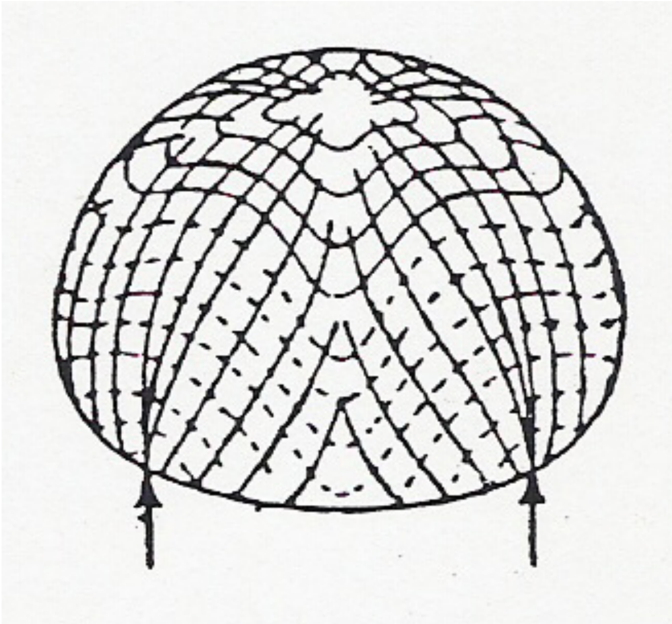
(Source: M. Farshad 1992)

Shells

- **Basic theory of shells**
  - **Effect of boundary/support conditions:**



(a) Stress trajectories in a dome with continuous support

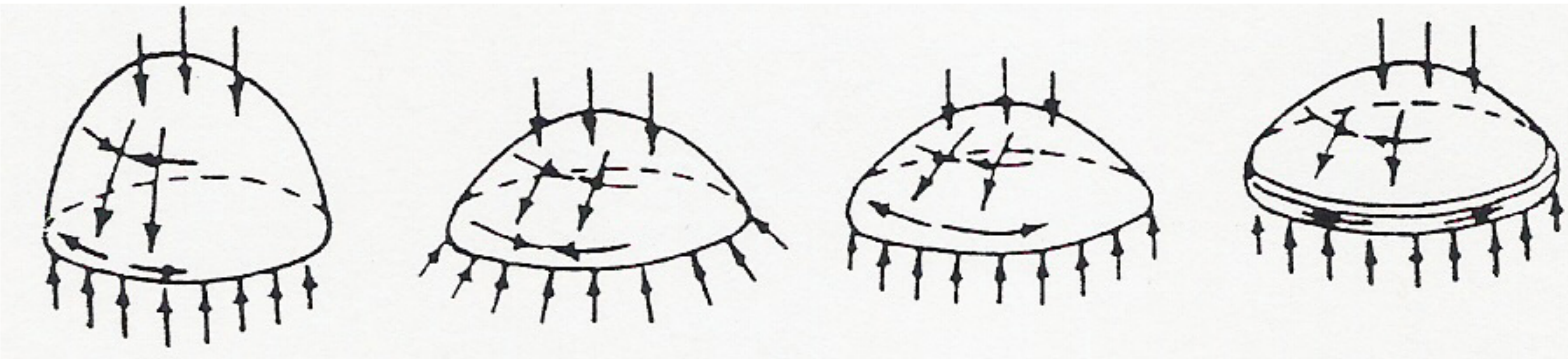


(b) Stress trajectories in a dome on discrete supports

(Source: M. Farshad 1992)

Shells

- **Basic theory of shells**
  - **Membrane behavior of axisymmetrically loaded domes:**



(a) High rise dome with roller (vertical) support

(b) A low rise dome with hinged (vertical and horizontal) support

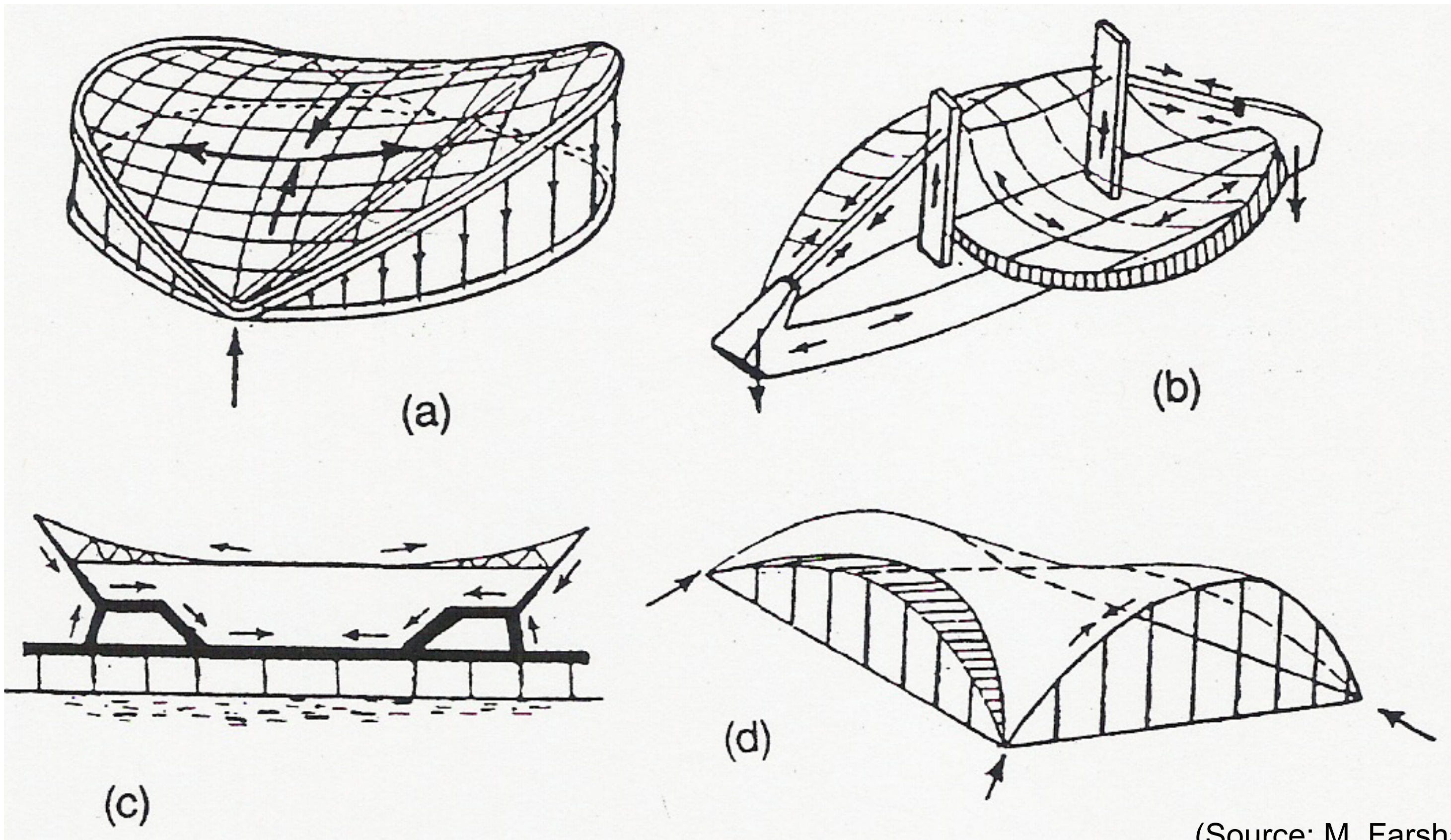
(c) A low rise dome with roller (vertical) support

(d) A low rise dome with roller (vertical) support and edge ring

(Source: M. Farshad 1992)

Shells

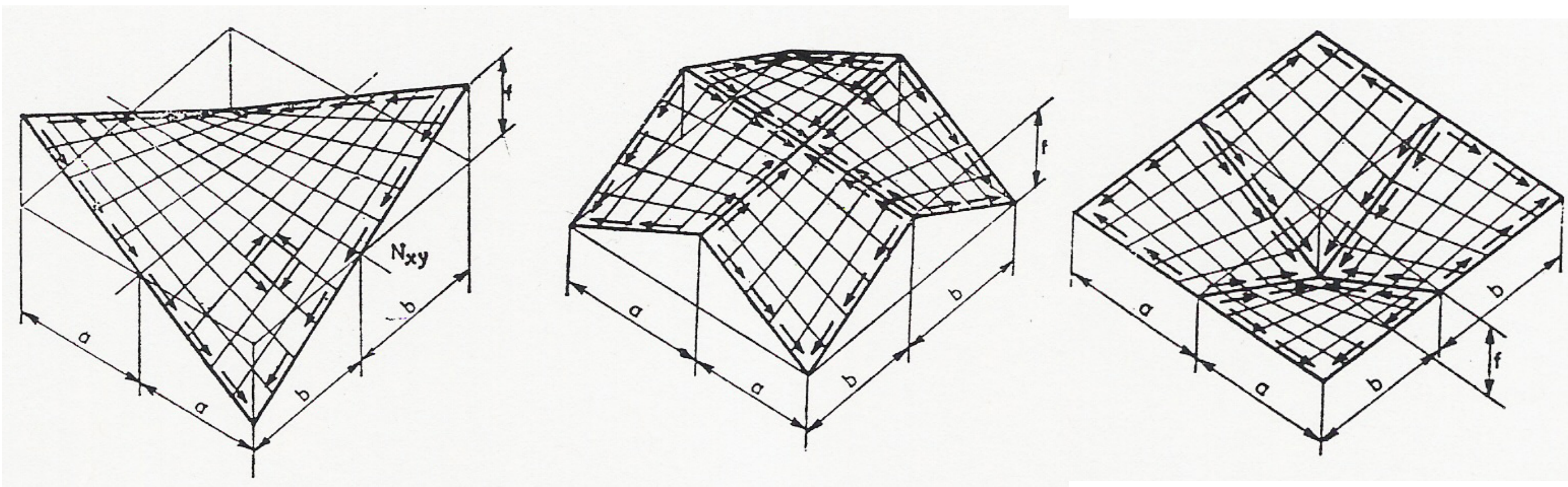
- **Basic theory of shells**
  - **Stress trajectories of some shell structures:**



(Source: M. Farshad 1992)

Shells

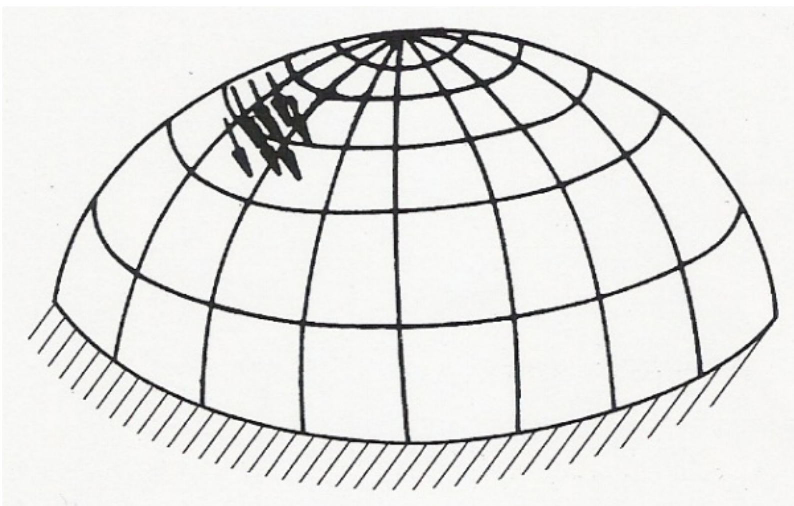
- **Basic theory of shells**
  - **Stress trajectories of some shell structures:**



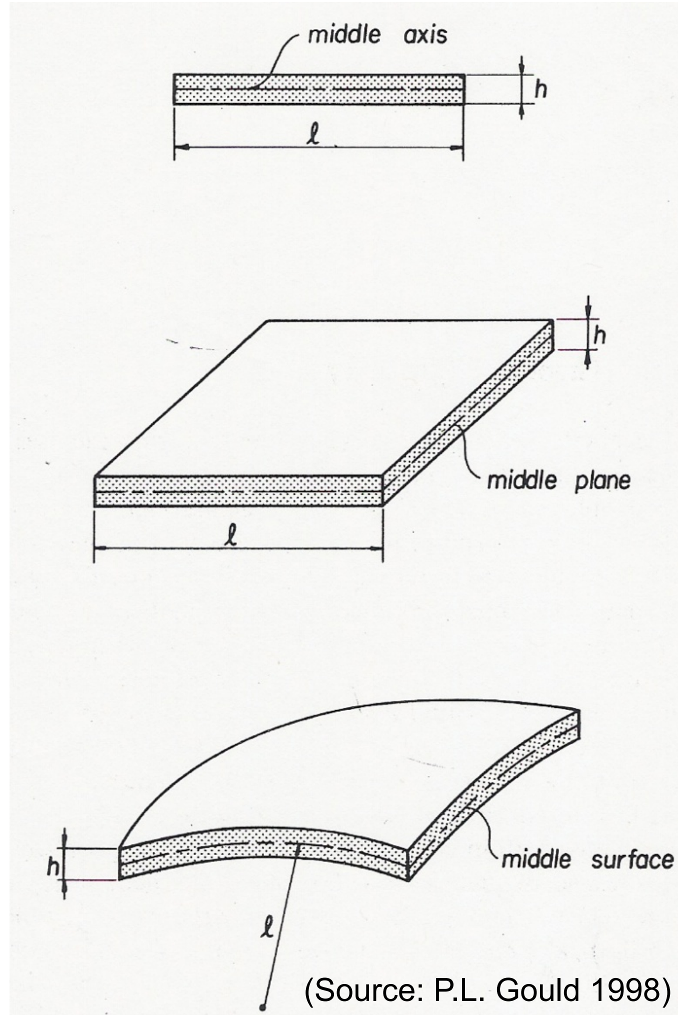
(Source: M. Farshad 1992)

Shells

- **Basic theory of thin shells**
  - **Definition:**
    - A thin shell is a curved slab whose thickness *h* is small compared with its other dimensions and compared with its principal radius of curvature.



(Source: A. Zingoni 1997)



(Source: P.L. Gould 1998)

Shells

- **Basic theory of thin shells**
  - **The Kirchhoff-Love Theory for Thin Shells** (Love 1888)
    - The shell thickness is negligibly small in comparison with the least radius of curvature of the shell mid-surface.
    - Strains and displacements that arise within the shells are small.
    - Straight lines that are normal to the mid-surface prior to deformation remain straight and normal to the middle surface during deformation, and experience no change in length. → Analogous to Navier's hypothesis for beams – Bernoulli-Euler theory for beams
    - The direct stress acting in the direction normal to the shell middle surface is negligible.
    - First-order approximation of shells

    → While usually convenient to use, the Kirchhoff-Love theory is strictly applicable to thin shells. It predicts incorrect behavior of shells near concentrated transverse loads or junctions.

Shells

- **Basic theory of thin shells**
  - **The Membrane Theory for Shells**
    - In some shells the stress couples are an order of magnitude smaller than the extensional and in-plane shear stress resultants.

      → The transverse shear stress resultants are similarly small and may be neglected in the force equilibrium.

      → Only valid for the shells whose one radius of curvature is finite.

    - This class of shells may achieve force equilibrium through the action of in-plane forces alone. The state of stress in the shell is completely determined by equations of equilibrium. → The shell is statically determinate.

    - The boundary conditions must also permit those shell edge displacements (translations and rotations) which are computed from the forces found by the membrane theory.

Shells

- **Examples of shell structures**

Fig. 1-3 (a) Pantheon, Rome, Italy, Dome Span = 43.4 m; Dome Rise = 21.6 m

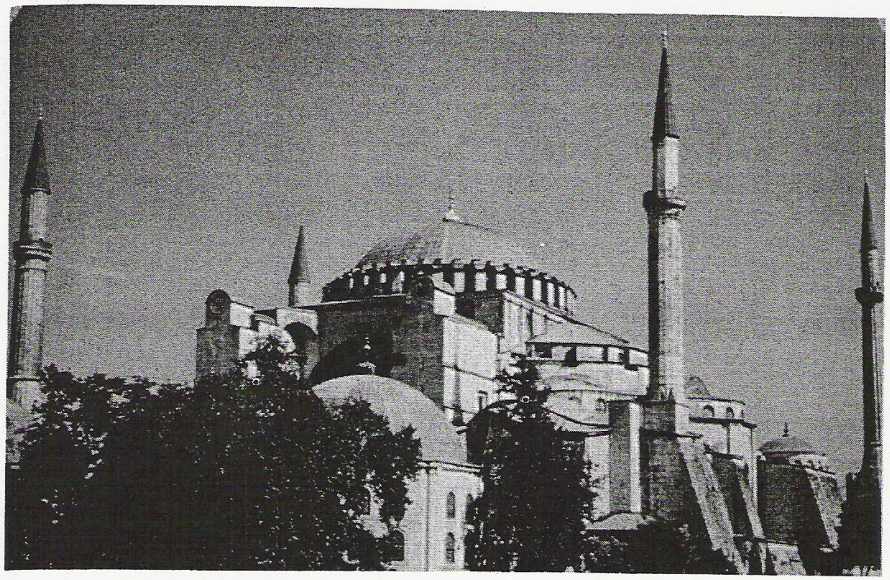


Fig. 1-4 Hagia Sophia, Istanbul, Turkey. Dome Span = 31.9 m; Dome Rise = 13.8 m (Courtesy Dr. I. Mungan)

(Source: P.L. Gould 1998)

Shells

- **Examples of shell structures**

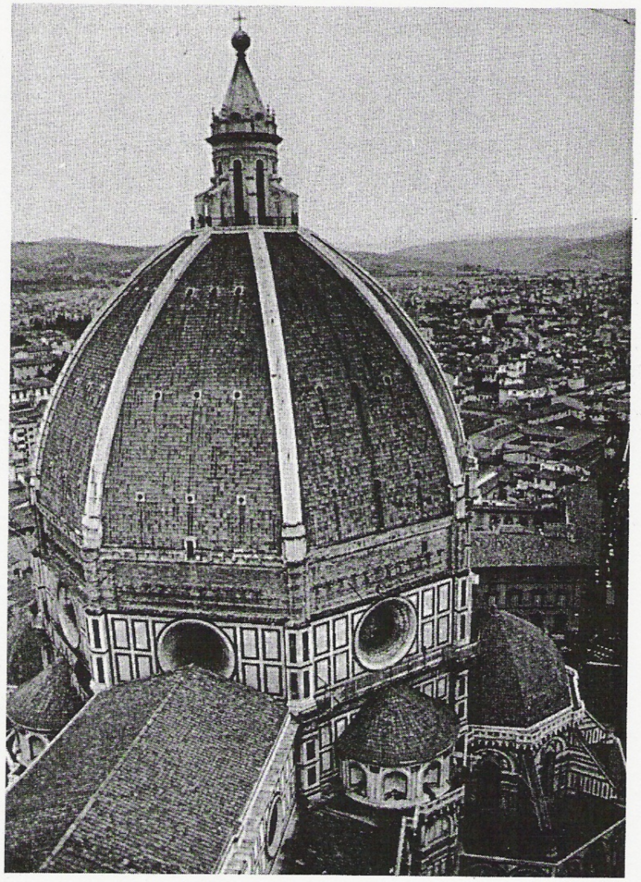


Fig. 1-6(a) S. Maria Del Fiore, Florence, Italy. Dome Span = 42.4 m; Dome Rise = 36.6 m

Fig. 1-7 St. Peter's, Rome, Italy. Dome Span = 41.6 m; Dome Rise = 35.1 m

(Source: P.L. Gould 1998)

Shells

- **Examples of shell structures**

Fig. 1-8 St. Paul's, London, England, Dome Span = 30.8 m; Dome Rise = 33.5 m

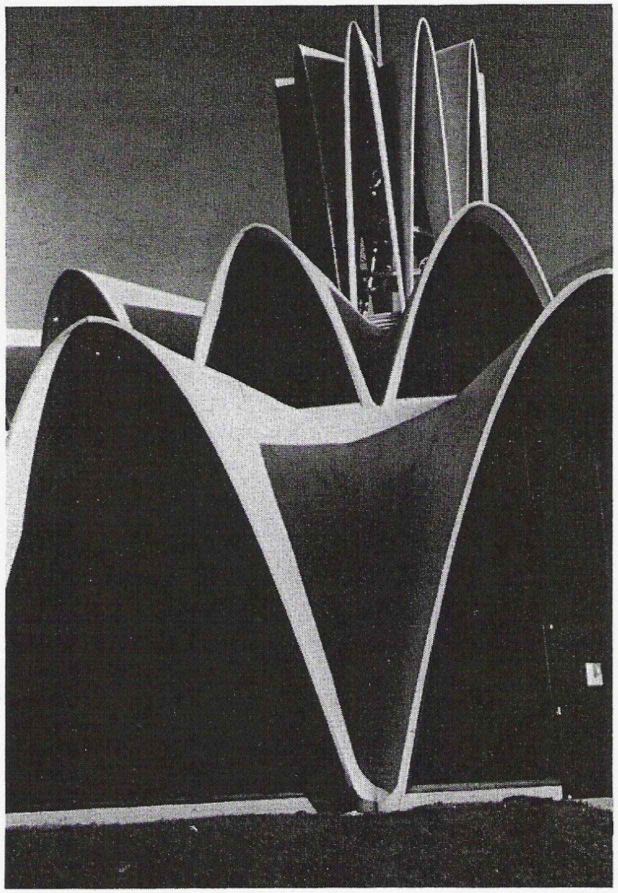


Fig. 2-8(g) Hyperboloid of Revolution, Planetarium, St. Louis, MO

(Source: P.L. Gould 1998)

Shells

- **Examples of shell structures**

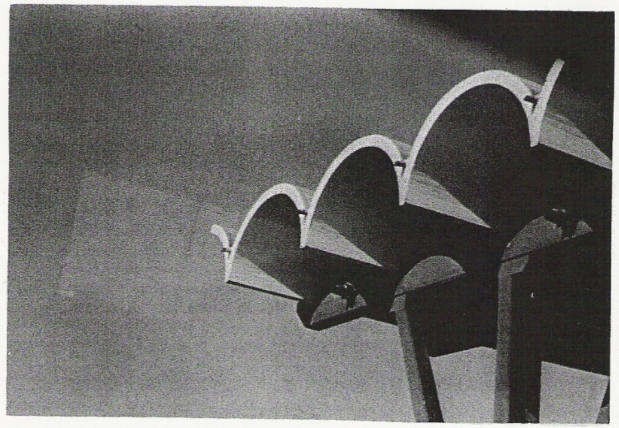


Fig. 2-8(h) Open Cylindrical Roof, Airport, Barcelona, Spain

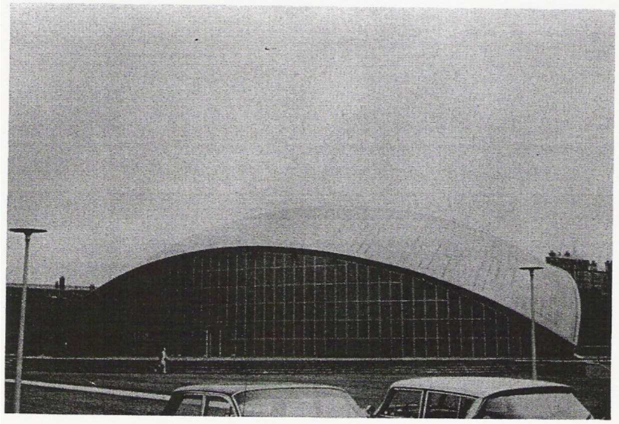


Fig. 2-8(i) Spherical Roof, Auditorium, Cambridge, MA

Fig. 2-8(j) Kingdome, Seattle, WA (Courtesy Dudley, Hardin & Yang, Inc.)

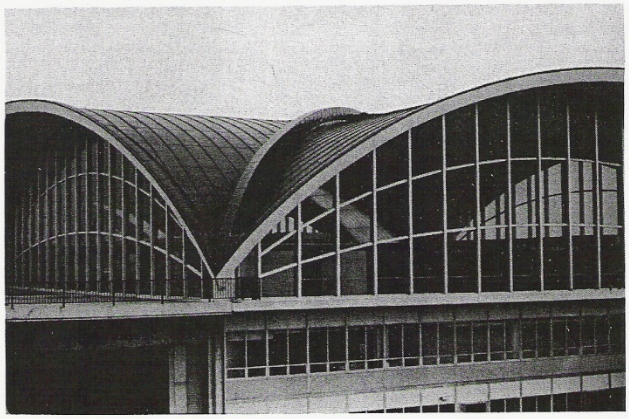


Fig. 2-8(k) Intersecting Barrel Shells, Airport, St. Louis, MO

(Source: P.L. Gould 1998)

Shells

- **Examples of shell structures**

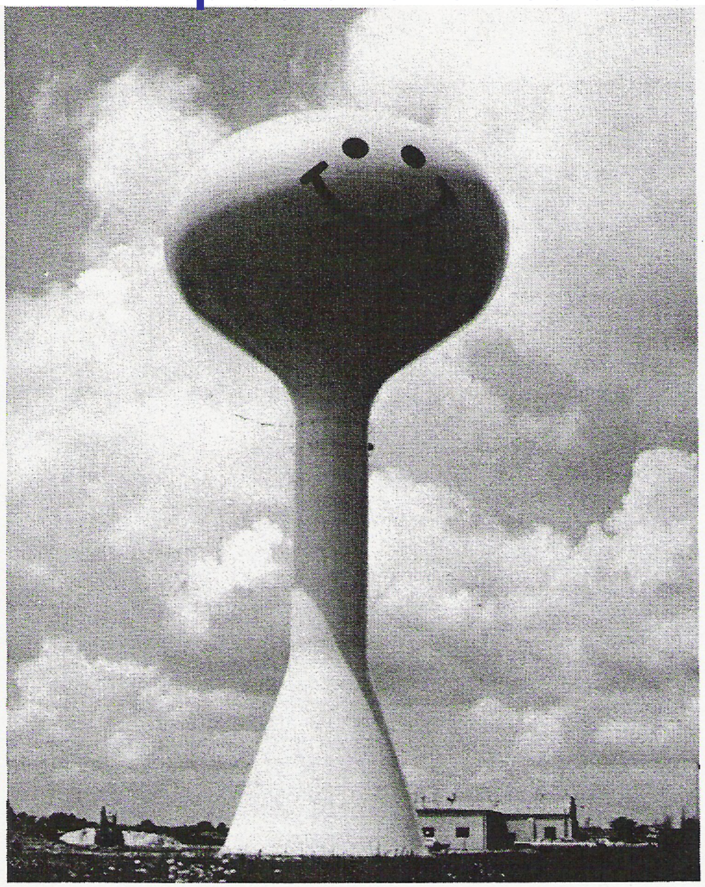


Fig. 2-8(s) Spheroidal Water Tower (Courtesy Chicago Bridge & Iron Co.)

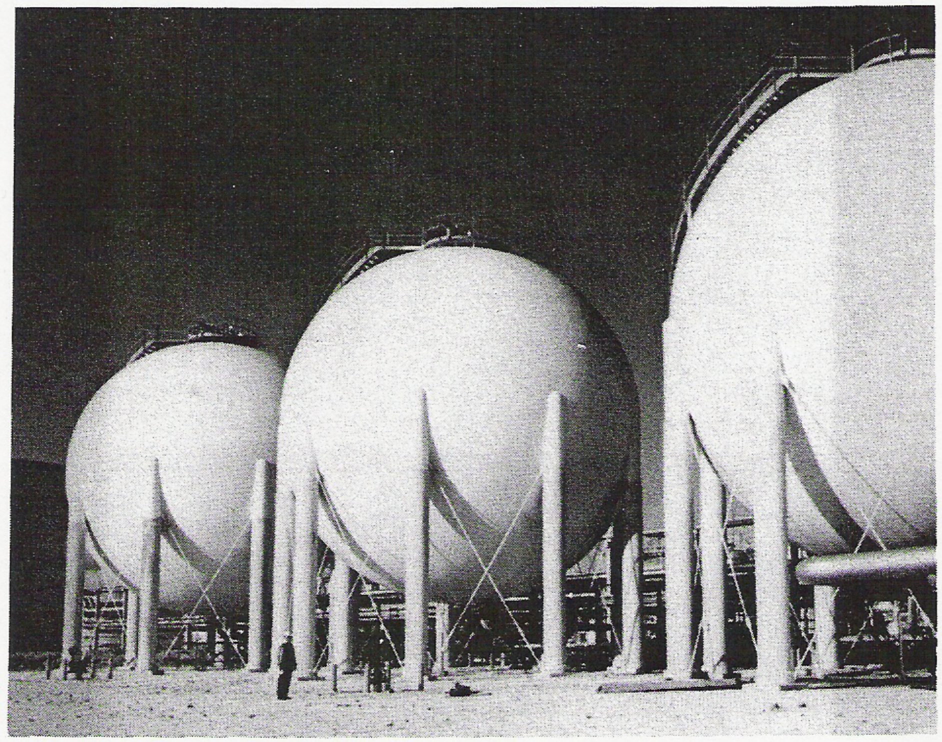
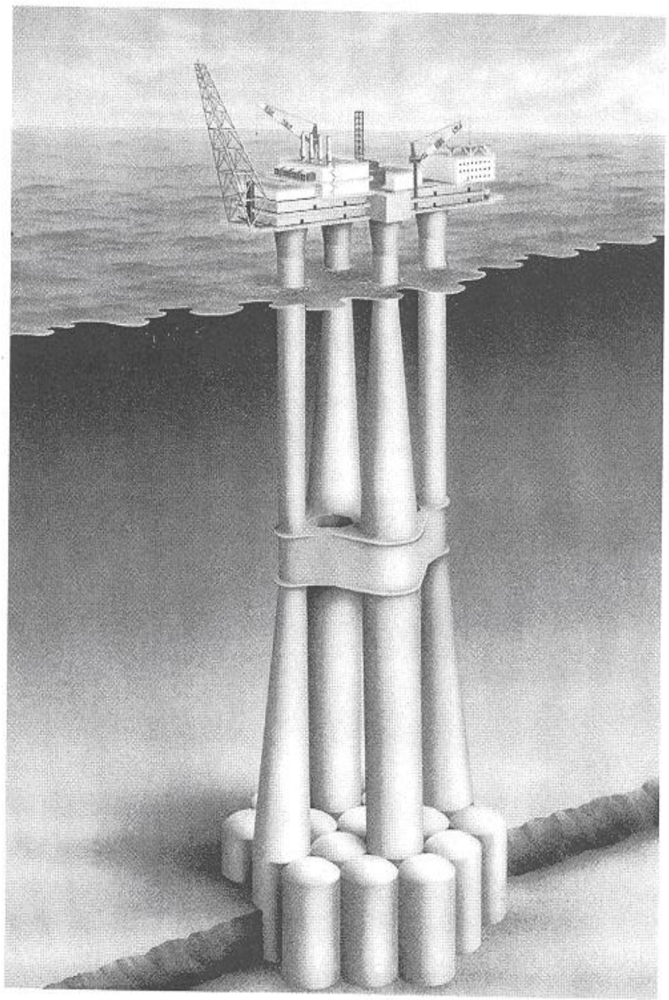


Fig. 2-8(t) Column-Supported Spherical Tanks (Courtesy Chicago Bridge & Iron Co.)

(Source: P.L. Gould 1998)

Shells

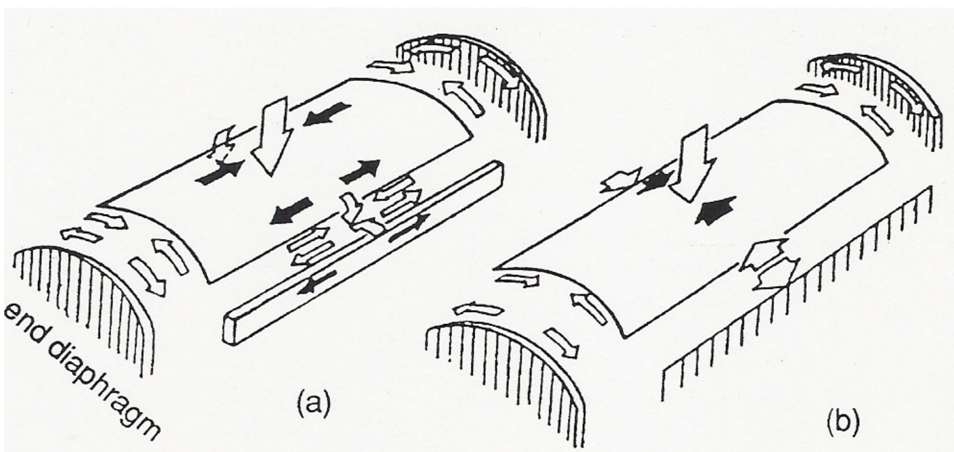


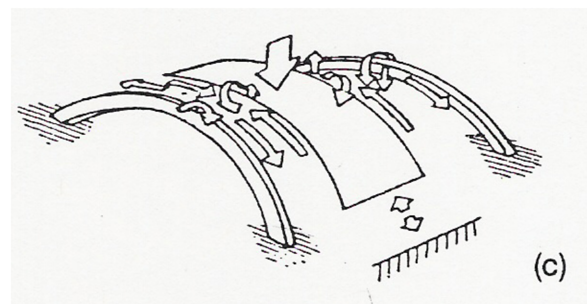
Troll A Platform, Norway

The 472-meter (1,548-foot) tall Troll A platform was towed to the offshore field in 1995, making it the largest structure humanity had ever moved at the time.

Shells

- **Analysis of Cylindrical Shells**
  - **Definition:**
    - A cylindrical shell can be defined as a curved slab taken form a full cylinder. The slab is bounded by two straight “longitudinal” edges parallel to the axis of the cylinder and by two curved transverse edges in planes perpendicular to the axis; the slab is curved in only one direction. The cylindrical shell is circular when the curvature is constant.
  - Effects of shell edges on the load carrying behavior

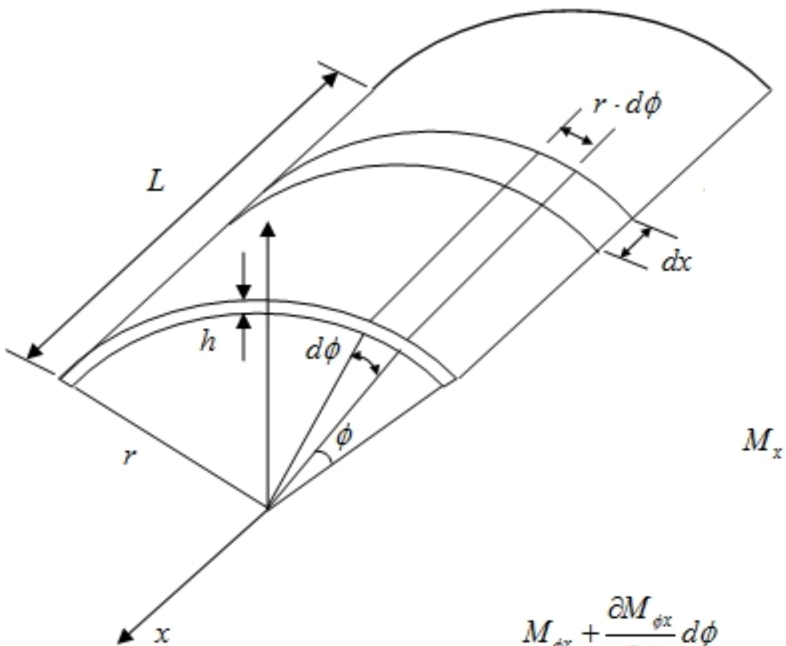




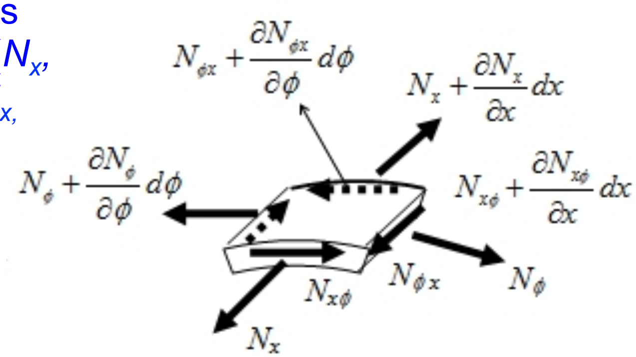
(Source: M. Farshad 1992)

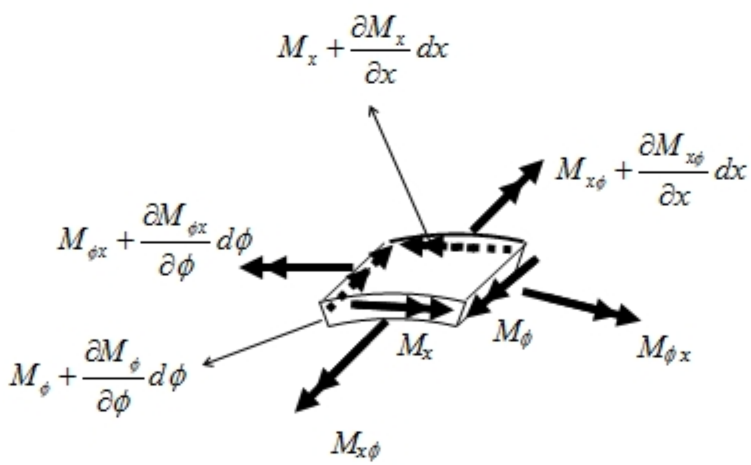
Shells

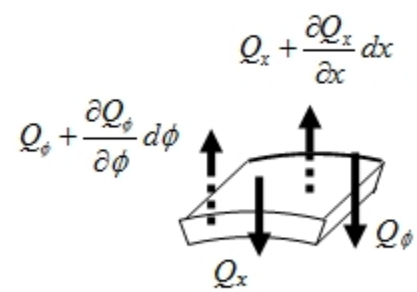
- **Analysis of Cylindrical Shells**
  - **Stress resultants and stress couples:**



• There are ten unknowns in the unit cell of shells; (N_x, N_f, N_{xf}, $N_{f\,x}$), (Q_x, Q_f), (M_x, M_f, M_{xf}, $M_{f\,x}$). → The problem is statically indeterminate.







Shells

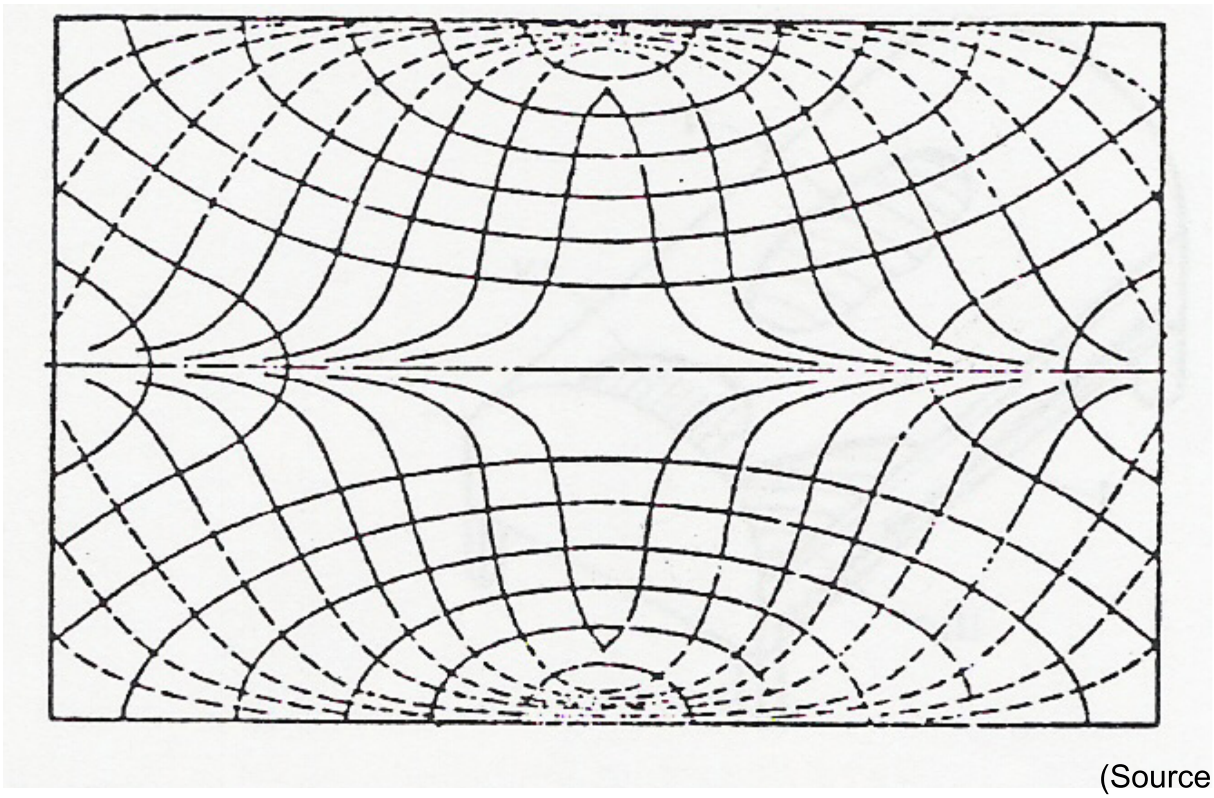
- **Analysis of Cylindrical Shells**
  - Long shells → $L/r \geq 2.5$
    - Line loads produce significant magnitudes of and , membrane forces become insignificant. Stresses can be estimated using the beam theory.
  - Intermediate shells → $0.5 \leq L/r < 2.5$
  - Short shells → $L/r < 0.5$
    - The line loads produce internal forces generally in the region near the longitudinal edge. Greater part of the shell behaves with membrane values.

  - Line loads: Forces applied along the free edge.

  - For long shells the stresses can be estimated closely by the beam theory (the shell is considered as a beam of a curved cross section between end supports). → Relative displacements within each transverse cross section are negligible.

Shells

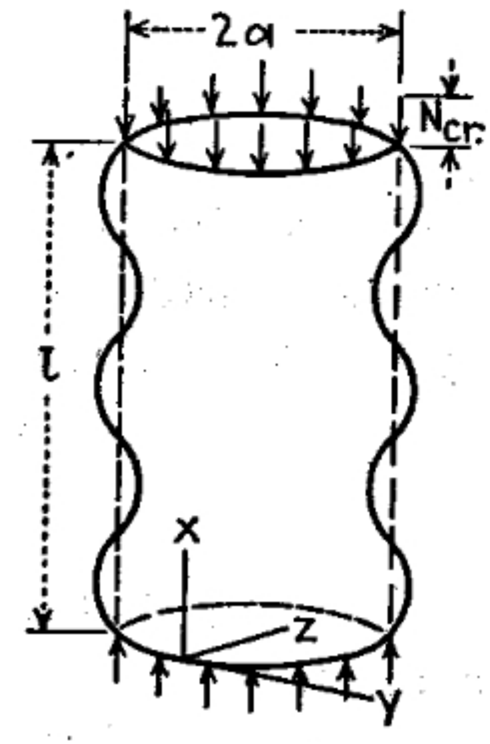
- **Analysis of Cylindrical Shells**
  - **Stress trajectories for a simply-supported cylindrical vault under uniform dead load**



(Source: M. Farshad 1992)

Shells

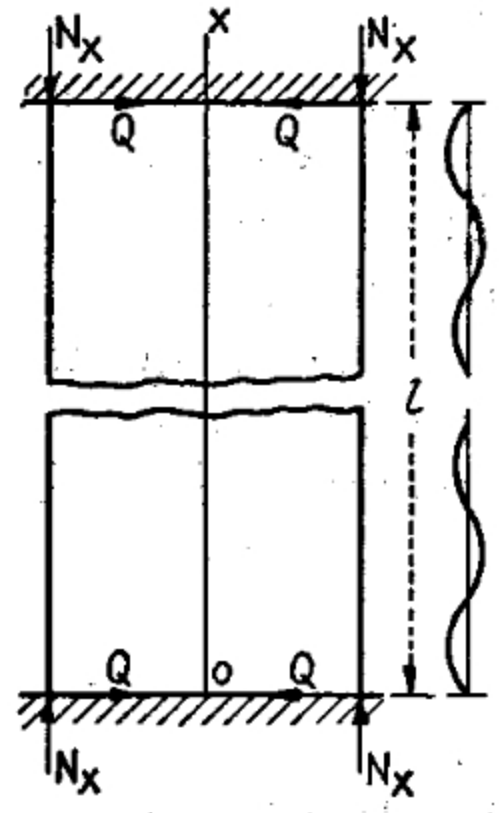
- **Buckling of Cylindrical Shells – Uniformly-distributed axial load**
  - Governing equation of symmetric buckling of a cylindrical shell
  - Radial displacements during buckling



(Source: Timoshenko and Gere 1961)

Shells

- **Buckling of Cylindrical Shells – Uniformly-distributed axial load**
  - Critical stress for thin shells
  - Critical stress for thick shells



(Source: Timoshenko and Gere 1961)

Shells

- **Buckling of Cylindrical Shells – Uniformly-distributed axial load**
  - Boundary conditions

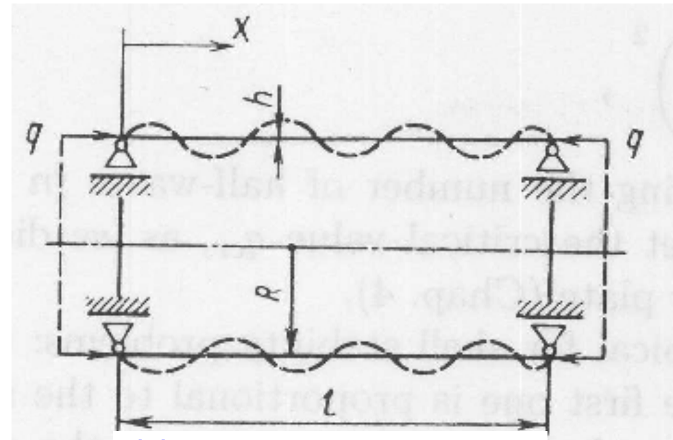
Edge condition	Prescribed d.o.f.	Natural condition
Clamped	$w = \theta_n = \theta_s = 0$	None
Simply supported	$w = 0$	$M_n = 0$
Free	None	$Q = M_n = M_{ns} = 0$

θ_n, M_n – rotation and moment normal to edge

θ_s, M_s– rotation and moment perpendicular to edge

Shells

- **Buckling of Cylindrical Shells – Uniformly-distributed axial load**

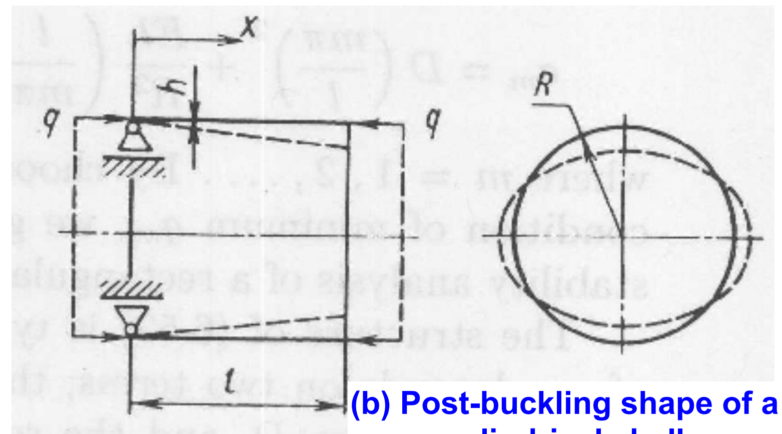


(a) Axisymmetric buckling

(Source: N.A. Alfutov 2000)

Shells

- **Buckling of Cylindrical Shells – Uniformly-distributed axial load**

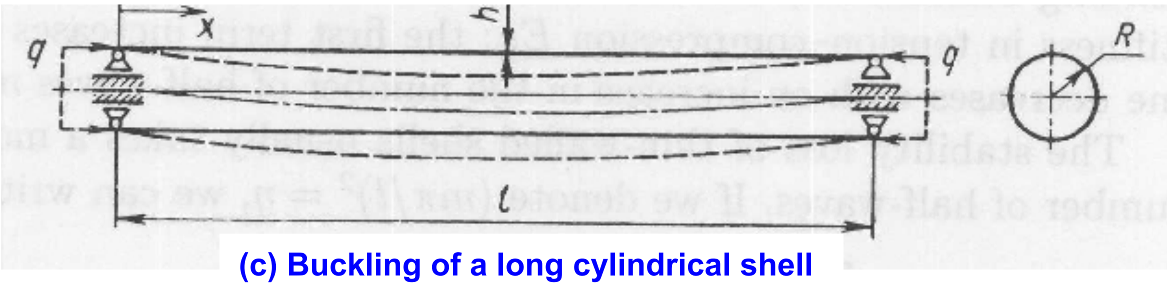


(b) Post-buckling shape of a cylindrical shell

(Source: N.A. Alfutov 2000)

Shells

- **Buckling of Cylindrical Shells – Uniformly-distributed axial load**

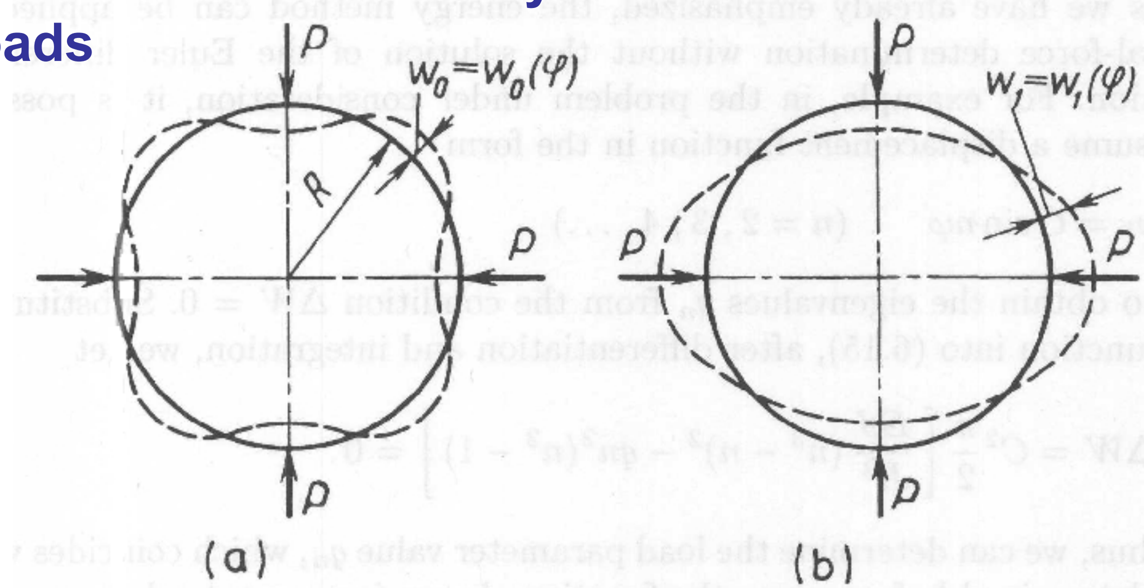


(c) Buckling of a long cylindrical shell

(Source: N.A. Alfutov 2000)

Shells

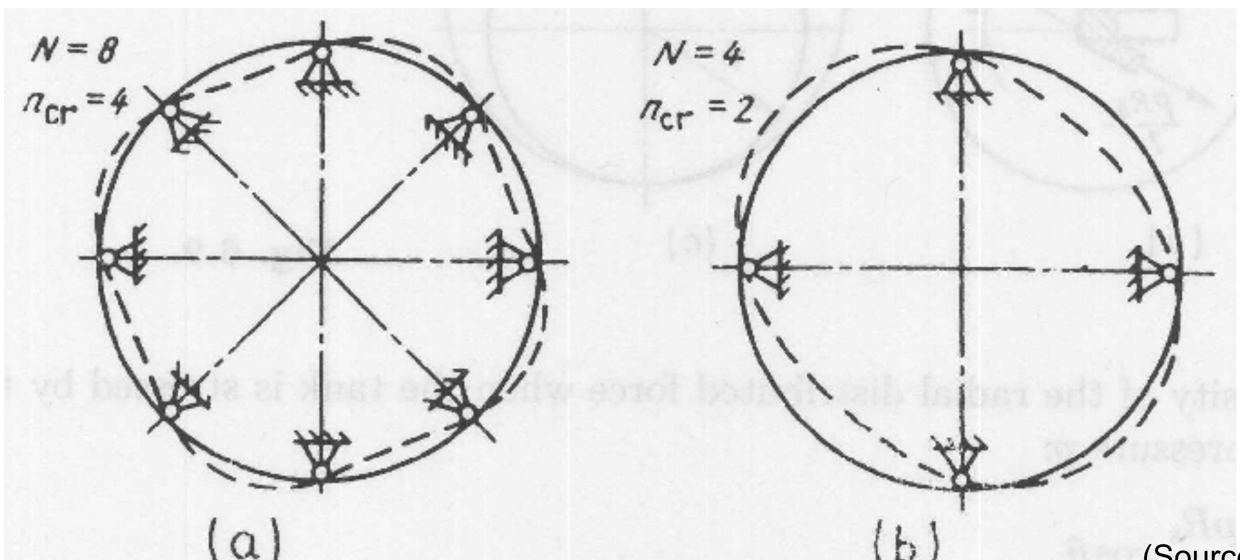
- **Buckling of Cylindrical Shells – Uniformly-distributed axial load and concentrated loads**



(Source: N.A. Alfutov 2000)

Shells

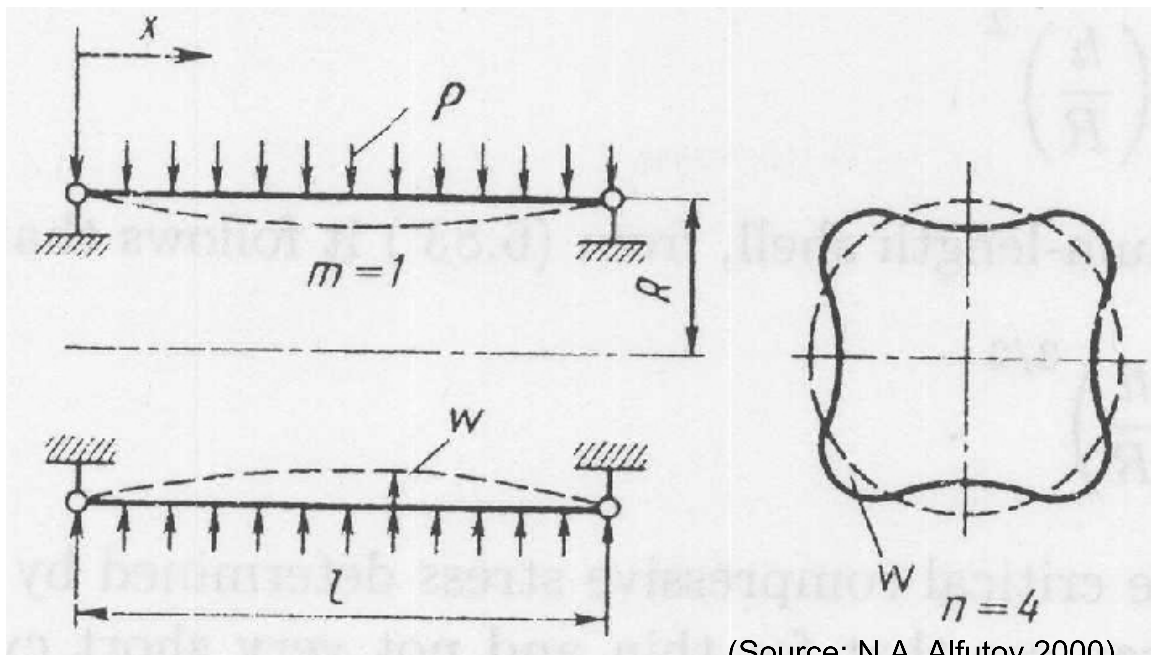
- **Buckling of Cylindrical Shells – Uniformly-distributed axial load and concentrated Loads**
  - When the shell has an even number of rigid supports which are uniformly distributed:



(Source: N.A. Alfutov 2000)

Shells

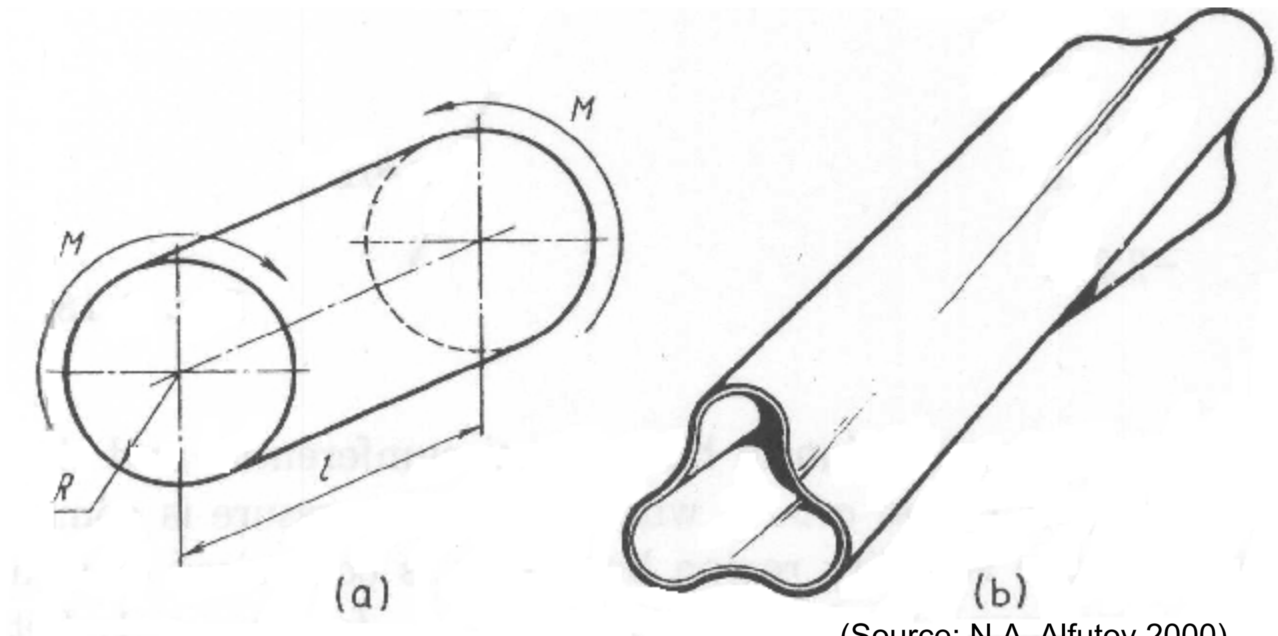
- **Buckling of Cylindrical Shells – Pure Bending**



(Source: N.A. Alfutov 2000)

Shells

- **Buckling of Cylindrical Shells – Torsion and transverse bending**



(Source: N.A. Alfutov 2000)

Advanced Topics

- **The Routh-Hurwitz Theorem**
  - The Hurwitz Polynomials
  - The Hurwitz Matrix
  - Theorem
  - Examples

Advanced Topics

- **The Routh-Hurwitz Theorem**
  - The Hurwitz Polynomials
  - The Hurwitz Matrix

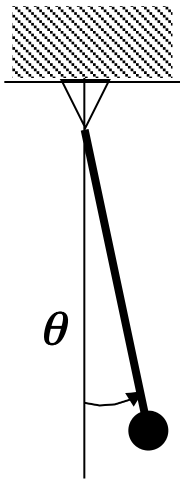
Advanced Topics

- **The Routh-Hurwitz Theorem**
  - Theorem
    - A necessary and sufficient condition for the n^{th} order polynomial to be a Hurwitz polynomial is that all of the principal minors Δ_1, Δ_2, … Δ_n of the Hurwitz matrix H to be positive.

→ $\Delta_n = a_n \Delta_{n-1} > 0$

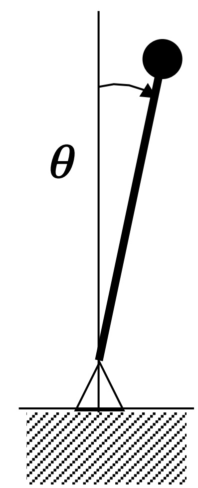
Advanced Topics

- **The Routh-Hurwitz Theorem**
  - Example #1: Damped pendulum problem



Advanced Topics

- **The Routh-Hurwitz Theorem**
  - Example #2: Inverted damped pendulum problem



Advanced Topics

- **The Lyapunov Theorems**
  - The Lyapunov Stability Theorem
    - For discrete systems with governing equations of the form

      $d\mathbf{x}/dt = \boldsymbol{X(x)}$,

      consider a real continuous function $\boldsymbol{V(x)}$ (generalized velocity function or Lyapunov functional) with following properties:
      - $\boldsymbol{V(x)}$ is positive (negative) definite if $\boldsymbol{V(x) > 0\ (< 0)}$ for all $\boldsymbol{x \neq 0}$ and $\boldsymbol{V(0) = 0}$.
      - $\boldsymbol{V(x)}$ is positive (negative) semi-definite if $\boldsymbol{V(x) \geq 0\ (\leq 0)}$ and it can vanish also for some $\boldsymbol{x \neq 0}$.
      - $\boldsymbol{V(x)}$ is indefinite if it can assume both positive and negative values in the domain of interest.

Advanced Topics

- **The Lyapunov Theorems**
  - The Lyapunov Stability Theorem
    - If there exists for the system a positive (negative) definite $V(x)$ whose total derivative $dV(x)/dt$ is negative (positive) semi-definite along every trajectory of $dx/dt = X(x)$, then the origin is *Lyapunov stable*.
    - If there exists for the system a positive (negative) definite $V(x)$ whose total derivative $dV(x)/dt$ is negative (positive) semi-definite along every trajectory of $dx/dt = X(x)$, then the trivial solution *asymptotically Lyapunov stable*.

    - If $V(x) > 0$ and $V(x) \leq 0$ →Lyapunov stable
    - If $V(x) < 0$ and $V(x) \geq 0$ →Lyapunov stable

Advanced Topics

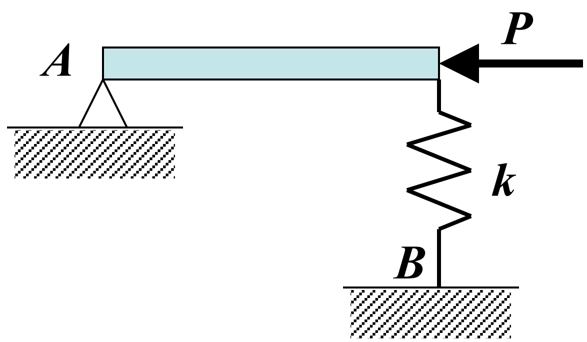
- **The Lyapunov Theorems**
  - The Lyapunov Instability Theorem
    - If there exists for the system a function $V(x)$ whose total derivative $dV(x)/dt$ is positive (negative) semi-definite along every trajectory of $dx/dt = X(x)$, and if the function itself can assume positive (negative) values for arbitrarily small values of x, then the trivial solution is *Lyapunov unstable*.

    - If $V(x) < 0$ and $V(x) \leq 0$ →Lyapunov unstable
    - If $V(x) > 0$ and $V(x) \geq 0$ →Lyapunov unstable

  - The use of Lyapunov's function is Lyapunov's Direct (Second) Method.

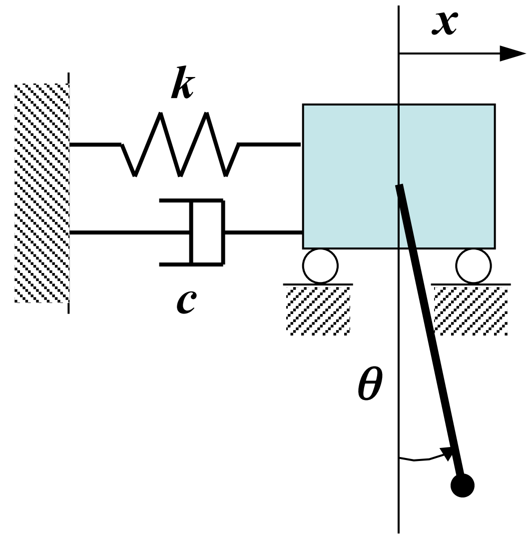
Advanced Topics

- **Application of the Lyapunov Stability Theorems**
  - Example #3 (Static Stability): Rigid bar-spring system



Advanced Topics

- **Application of the Lyapunov Stability Theorems**
  - Example #4 (Dynamic Stability): Spring-mass-damper system with a pendulum



Advanced Topics

- **Application of the Lyapunov Stability Theorems**
  - Example #5 (Dynamic Stability): The van del Pol equation

Advanced Topics

- **Application of the Lyapunov Stability Theorems**
  - Example #6 (Dynamic Stability): Turbulent flow in a channel

Advanced Topics

- **The Lyapunov Stability Theorems**
  - Guideline for deriving Lyapunov's functionals:
    - For mass-dependent systems,
    - For mass-independent systems,

Summary

- In the linear theory of shells the governing equations may be rendered hyperbolic as a consequence of geometrical properties of the shell surface.
- Although shells are efficient structures for loading, the rapid change in geometry after buckling and consequent decrease of load capacity can lead to catastrophic collapse.
- Boundary/support conditions can significantly change the stress distribution within shell structures.
- The Kirchhoff-Love theory is usually used for thin shells subjected to uniformly-distributed loads.
- Lyapunov’s Direct (Second) Method is based on Dirichlet’s proof of Lagrange’s Theorem on the stability of equilibrium of a system.
- Lyapunov’s functionals have a close relationship with energy functions.
- The way to derive Lyapunov’s functionals is not strictly formulated; sometimes the Lyapunov’s functional for a system has not been defined.
- The Lyapunov stability theorems can be applied to both static and dynamic stability problems.

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