Math Methods I - curvilinear coords

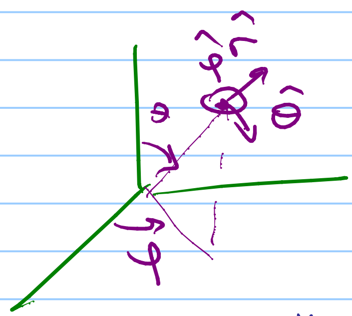
Note Title 9/27/2014

① Cartesian coordinates provide a limited coordinate choice

Notably, all unit vectors have dimension of length and their direction is independent of the location $\vec{r}$

In general, this is not true.

Consider, for example spherical coordinates:



r, θ, φ : (note order)

the direction of the unit vectors depends on location!

Vector operations therefore can only be defined for two vectors @ the same point. the form of vector operations is unchanged:

$$\vec{V}_1 = V_{1r}\hat{r} + V_{1\theta}\hat{\theta} + V_{1\varphi}\hat{\varphi}$$

$$\vec{V}_2 = V_{2r}\hat{r} + V_{2\theta}\hat{\theta} + V_{2\varphi}\hat{\varphi}$$

$$\vec{V}_1 \cdot \vec{V}_2 = V_{1r}V_{2r} + V_{1\theta}V_{2\theta} + V_{1\varphi}V_{2\varphi}$$

Consider now the element of length:

$$dr^2 = dx^2 + dy^2 + dz^2 = [dx_1^2 + dx_2^2 + dx_3^2]$$

$$= \sum_{i=1}^{3} dx_i^2 = \sum_{i=1}^{3} dx_i\, dx_i =$$

$$= \sum_{i,j,k} \frac{\partial x_i}{\partial q_j} dq_j \frac{\partial x_i}{\partial q_k} dq_k = \sum_{jk} \overbrace{\boxed{\left(\sum_i \frac{\partial x_i}{\partial q_j}\frac{\partial x_i}{\partial q_k}\right)}}^{g_{jk}} dq_j\, dq_k$$

$$= g_{jk}\, dq_j\, dq_k$$

if $\hat{\epsilon}_i \cdot \hat{\epsilon}_k = \delta_{ik}$ - the coordinate system is called orthogonal

in orthogonal systems $\boxed{g_{jk} = h_j^2\, \delta_{jk}}$

$$dr_i = h_i\, dq_i$$

$$\frac{\partial \vec{r}}{\partial q_i} = h_i \hat{e}_i \ [= \vec{\epsilon}_i\ ]$$

---

② differential operators are defined as:

$$\vec{\nabla} \varphi = \lim_{\int d\tau \to 0} \frac{\int (d\vec{\sigma})\, \varphi}{\int d\tau}$$

$$\vec{\nabla} \cdot \vec{V} = \lim_{\int d\tau \to 0} \frac{\int d\vec{\sigma} \cdot \vec{V}}{\int d\tau}$$

$$\vec{\nabla} \times \vec{V} = \lim_{\int d\tau \to 0} \frac{\int d\vec{\sigma} \times \vec{V}}{\int d\tau}$$

$$\vec{\nabla} \varphi = \sum_i \frac{1}{h_i} \frac{\partial \varphi}{\partial q_i}\, \hat{e}_i$$

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1}(h_2 h_3 V_1) + \frac{\partial}{\partial q_2}(h_1 h_3 V_2) + \frac{\partial}{\partial q_3}(h_1 h_2 V_3) \right]$$

$$\vec{\nabla} \times \vec{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

Example : Spherical coords:

$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases}$$

$$h_1^2 = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = \overbrace{\sin^2\theta \cos^2\varphi + \sin^2\theta \sin^2\varphi}^{\sin^2\theta} + \cos^2\theta = 1$$

$$h_2^2 = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = \underbrace{r^2 \cos^2\theta \cos^2\varphi + r^2 \cos^2\theta \sin^2\varphi}_{r^2\cos^2\theta} + r^2 \sin^2\theta = r^2$$

$$h_3^2 = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2 = r^2 \sin^2\theta \sin^2\varphi + r^2 \sin^2\theta \cos^2\varphi =$$

$$= r^2 \sin^2\theta$$

$$\vec{\nabla}\psi = \lim_{\int d\tau\to 0} \frac{\int \psi\, d\vec{\sigma}}{\int d\tau} =$$

$+\dots$

$$= \lim_{\int d\tau\to 0} \frac{\left[\left\{\psi h_2 h_3 \hat{q}_1 + \frac{\partial}{\partial q_1}(\hat{q}_1 \psi h_2 h_3)\, dq_1\right\} - \psi\cdot h_2 h_3\right] dq_2\, dq_3}{\int h_1 h_2 h_3\, dq_1 dq_2 dq_3}$$

$$= \lim_{\int d\tau\to 0} \int \frac{\left[\frac{\partial}{\partial q_1}(h_2 h_3 \hat{q}_1 \psi) + \frac{\partial}{\partial q_2}(\hat{q}_2 h_1 h_3 \psi) + \frac{\partial}{\partial q_3}(\hat{q}_3 h_1 h_2 \psi)\right] d^3 q_i}{\int h_1 h_2 h_3\, dq_1 dq_2 dq_3}$$

$$= \frac{1}{h_1 h_2 h_3}\Big\{ h_2 h_3 \hat{q}_1 \frac{\partial \psi}{\partial q_1} + \hat{q}_1 \psi h_2 \frac{\partial h_3}{\partial q_1} + \hat{q}_1 \psi h_3 \frac{\partial h_2}{\partial q_1} +$$

$$+ \psi h_2 h_3 \frac{\partial \hat{q}_1}{\partial q_1} +$$

$$+ h_1 h_3 \hat{q}_2 \frac{\partial \psi}{\partial q_2} + \psi \hat{q}_2 h_1 \frac{\partial h_3}{\partial q_2} + \psi \hat{q}_2 h_3 \frac{\partial h_1}{\partial q_2} + \psi h_1 h_3 \frac{\partial \hat{q}_2}{\partial q_2}$$

$$+ h_1 h_2 \hat{q}_3 \frac{\partial \psi}{\partial q_3} + \psi \hat{q}_3 h_1 \frac{\partial h_2}{\partial q_3} + \psi \hat{q}_3 h_2 \frac{\partial h_1}{\partial q_3} + \psi h_1 h_2 \frac{\partial \hat{q}_3}{\partial q_3} \Big\}$$

$$= \left\{ \hat{q}_1 \frac{\partial \psi}{h_1 \partial q_1} + \hat{q}_2 \frac{\partial \psi}{h_2 \partial q_2} + \frac{\hat{q}_3}{h_3} \frac{\partial \psi}{\partial q_3} \right\} +$$

$$+ \hat{q}\psi \left\{ \cancel{h_2 \frac{\partial h_3}{\partial q_1}} + \cancel{h_3 \frac{\partial h_2}{\partial q_1}} - \cancel{h_1 h_3 \frac{1}{h_1} \frac{\partial h_2}{\partial q_1}} - \cancel{h_1 h_2 \frac{1}{h_1} \frac{\partial h_3}{\partial q_1}} \right\} + \dots$$

3 Tensors

As shown earlier, in curvilinear coords components of space-like vectors

$$\boxed{A^{j'} = \sum_i \frac{\partial q^{j'}}{\partial q^i} A^i} \leftarrow \text{contravariant vector}$$

transform differently from components of gradient-like vectors:

$$\boxed{B_{j'} = \sum_i \frac{\partial q^i}{\partial q^{j'}} B_i} \leftarrow \text{covariant vector}$$

Similarly, we can define contravariant, covariant, and mixed tensors of rank n that transform according to

all coords are contravariant

$$T'^{\;\;\;\;i_3 \ldots i_N}_{i_1 i_2} = \sum_{j_1, \ldots, j_N} \frac{\partial q^{j_1}}{\partial q^{i_1'}} \frac{\partial q^{j_2}}{\partial q^{i_2'}} \frac{\partial q^{i_3'}}{\partial q^{j_3}} \cdots \frac{\partial q^{i_N'}}{\partial q^{j_N}} T_{j_1 j_2}^{\;\;\;\;j_3 \ldots j_N}$$

Note: summation is always over repeated "top" & "bottom" indices

---

δ_i^j – mixed 2nd rank tensor:

$$\delta_i'^{\,j} = \frac{\partial q'^j}{\partial q^k} \frac{\partial q^\ell}{\partial q^{i'}} \delta^k_\ell = \frac{\partial q'^j}{\partial q^k} \frac{\partial q^k}{\partial q'^i} = \frac{\partial q'^j}{\partial q'^i} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

(isotropic tensor)

Similarly, ε_{ijk} - isotropic 3-rd rank covariant pseudo-tensor. Can be used to provide correspondence between a pseudo-vector & 2-nd rank tensor

$$V_i = \frac{1}{3}\varepsilon_{ijk} C^{jk} ; \quad \text{Note: } C^{jk} = -C^{kj} -$$

has 3 independ. comp. in $\mathbb{R}^3$

Quotient rule: if relation

$$A_{\dots} = C^{\dots} B_{\dots}$$ works in all coord systems & A, B - tensors, then C - tensor

basis vectors:

$$\vec{\varepsilon}_i = \frac{\partial x}{\partial q^i} \hat{e}_x + \frac{\partial y}{\partial q^i} \hat{e}_y + \frac{\partial z}{\partial q^i} \hat{e}_z$$

then:

$$\vec{A} = A^i \vec{\varepsilon}_i$$

$$ds^2 = dr^2 = d\vec{r} \cdot d\vec{r} = (dq^i \vec{\varepsilon}_i) \cdot dq^j \vec{\varepsilon}_j =$$

$$= g_{ij}\, dq^i dq^j, \text{ with}$$

$$g_{ij} = \vec{\varepsilon}_i \cdot \vec{\varepsilon}_j$$ - metric tensor

we can define

$$g^{ij} \Leftarrow g^{ij} g_{jk} = \delta^i_k$$

and then use g^{ij} to define:

$$\vec{\varepsilon}^{\,j} = g^{ij}\vec{\varepsilon}_i$$

Explicitly:

$$\vec{\varepsilon}^{\,j} = \frac{\partial q^j}{\partial x}\hat{e}_x + \frac{\partial q^j}{\partial y}\hat{e}_y + \frac{\partial q^j}{\partial z}\hat{e}_z$$

$$g^{ik} = \vec{\varepsilon}^{\,i}\cdot\vec{\varepsilon}^{\,k}$$

Note:

$$\vec{A} = A^i\vec{\varepsilon}_i = A^k\delta^i_k\vec{\varepsilon}_i = A^k g_{lk} g^{il}\vec{\varepsilon}_i =$$

$= A_l\vec{\varepsilon}^{\,l}$ ← same vector; two different component representations

---

Covariant derivatives:

$$\vec{V} = V^i\vec{\varepsilon}_i$$

$$\frac{\partial\vec{V}}{\partial q^k} = \frac{\partial V^i}{\partial q^k}\vec{\varepsilon}_i + V^i\boxed{\frac{\partial\vec{\varepsilon}_i}{\partial q^k}}$$

same vector with components:

$$\frac{\partial\vec{\varepsilon}_i}{\partial q^k} = \Gamma^j_{ik}\vec{\varepsilon}_j \;;\; \Gamma^j_{ik} = \frac{\partial\vec{\varepsilon}_i}{\partial q^k}\cdot\vec{\varepsilon}_j$$

$$\frac{\partial\vec{V}}{\partial q^k} = \frac{\partial V^i}{\partial q^k}\vec{\varepsilon}_i + \Gamma^j_{ik}\vec{\varepsilon}_j V^i =$$

$$= \underbrace{\left(\frac{\partial V^i}{\partial q^k} + \Gamma^i_{jk}V^j\right)}_{i\text{-th component}}\vec{\varepsilon}_i$$

$$\rightarrow \quad V^i_{;k} = \frac{\partial V^i}{\partial q^k} + \Gamma^i_{jk} V^j$$

full differential:

$$d\vec{V} = \frac{\partial \vec{V}}{\partial q^k}\, dq^k = \underbrace{V^i_{;k}}_{\substack{\text{mixed} \\ \text{2-nd rank tensor}}} \overbrace{dq^k}^{\text{contravariant}} \underbrace{\vec{\varepsilon}_i}_{\text{covariant}}$$

---

differential operators:

$$\vec{\nabla}\psi = \frac{\partial \psi}{\partial q^i}\, \vec{\varepsilon}^{\,i}$$

$$\vec{\nabla} \cdot \vec{V} = V^i_{;i}$$