Note Title 10/19/2014

Vectors in function spaces

$$f(s) = \sum_i a_i \varphi_i(s)$$

components "basis functions"

Scalar product:

$$\langle f | g \rangle = \int_a^b f^*(s)\, g(s)\, w(s)\, ds$$

Other operations: addition, multiplication by a scalar

In orthonormal basis:

$$(\varphi_i | \varphi_j) = \delta_{ij}$$

$$f(s) = \sum_i a_i \varphi_i(s) \quad \left\{ \varphi_i(s) \right.$$

$$\langle \varphi_j | f(s) \rangle = a_j \, \|\varphi_j\|^2$$

$$\text{if } f(s) = \delta(s - s_0) \Rightarrow a_j = \varphi_j^*(s_0)$$

$$\delta(s - s_0) = \sum_i \frac{\varphi_i(s)\, \varphi_i^*(s_0)}{\|\varphi_i\|^2}$$

Orthonormalisation:

Start with φ_i

$$\varphi_1 \to \tilde{\varphi}_1 = \frac{\varphi_1}{\|\varphi_1\|}$$

$$\varphi_2 = \psi_2 + a_{21}\varphi_1$$

$$\langle \varphi_1 | \varphi_2 \rangle = \langle \varphi_1 | \psi_2 \rangle + a_{21}\|\varphi_1\|^2 = 0$$

$$a_{21} = -\frac{\langle \varphi_1 | \psi_2 \rangle}{\|\varphi_1\|^2} \Rightarrow$$

$$\varphi_m = \psi_m + \sum_{i=1}^{m-1} a_{mi}\varphi_i$$

$$\langle \varphi_j | \varphi_m \rangle = \langle \varphi_j | \psi_m \rangle + a_{mj}\|\varphi_j\|^2 = 0$$

$$\Rightarrow a_{mj} = -\frac{\langle \varphi_j | \psi_m \rangle}{\|\varphi_j\|^2}$$

Linear operators:

$$(A+B)f = Af + Bf \; ; \; A(f+g) = Af + Ag$$

$$Ak = kA \quad (k = \text{const})$$

Identity operator: $If = f$

Adjoint operator: $\langle f | Ag \rangle = \langle A^{+}f | g \rangle$

if $A^{+} = A$ - A - self - Adjoint (Hermitian)

if $A^{+} = A^{-1}$ - A - unitary

if A = real + unitary $\Rightarrow$ orthogonal

if A - linear operator, φ_j - orthonormal basis $\Rightarrow$

$$A\varphi_m = \sum_n a_{nm}\varphi_n$$

then: $$A\psi = \sum_m b_m \varphi_m$$

but: $A\psi = A\sum_i c_i \varphi_i = \sum_{i,n} a_{ni} c_i \varphi_n$

$$\Rightarrow b_n = a_{ni} c_i$$

matrix corresponding to A

$\Rightarrow$ can introduce functions of operators

Note that $\underbrace{\left[\sum_\nu |\varphi_\nu\rangle\langle\varphi_\nu|\right]}_{I}\psi = \sum_\nu b_\nu |\varphi_\nu\rangle = \psi$

$$A = \sum_{\nu,\mu} |\varphi_\nu\rangle\langle\varphi_\nu|A|\varphi_\mu\rangle\langle\varphi_\mu| = \sum_{\nu,\mu} |\varphi_\nu\rangle a_{\nu\mu} \langle\varphi_\mu|$$

Now,

$$\langle\chi|A|\psi\rangle = \langle A^\dagger\chi|\psi\rangle = \langle\psi|A^\dagger|\chi\rangle^*$$

$$\Rightarrow \langle\psi|A^\dagger|\chi\rangle = \langle\chi|A|\psi\rangle^* =$$

$$= \left[\sum_{\nu,\mu}\langle\chi|\varphi_\nu\rangle a_{\nu\mu}\langle\varphi_\mu|\psi\rangle\right]^* =$$

$$= \sum_{\nu,\mu} \underbrace{\langle\psi|\varphi_\mu\rangle a^*_{\nu\mu} \langle\varphi_\nu|\chi\rangle}_{} =$$

$A^\dagger \Rightarrow$ matrix of adjoint operator is Hermitian conj of the matrix of initial operator

$$\Rightarrow A = A^\dagger \Leftrightarrow a_{\nu\mu} = (a^*_{\nu\mu})^T = a^*_{\mu\nu}$$

Unitary operators:

Consider $\psi = \sum_i c_i \varphi_i = \sum_i |\varphi_i\rangle\langle\varphi_i|\,|\varphi\rangle$

changing the basis:

$$\varphi_i = \sum_j u_{ji}\, \varphi_j' = \sum_j \underbrace{\langle \varphi_j' | \varphi_i \rangle}_{u_{ji}} \varphi_j'$$

$$\Rightarrow \psi = \sum_i c_i \varphi_i = \sum_{i,j} c_i\, u_{ji}\, \varphi_j' = \sum u_{ji}\, c_i\, \varphi_j' = \sum_j c_j' \varphi_j'$$

$$\Rightarrow c_j' = u_{ji}\, c_i$$

inverse transformation: $c_i = (u^{-1})_{ij}\, c_j'$

where $(u^{-1})_{ij} = \langle \varphi_i | \varphi_j' \rangle = u_{ji}^*$

$$\Rightarrow u^{-1} = u^{+} \quad (u\text{-real} \Rightarrow u\text{-orthogonal})$$

transformation of operators:

$$A = |\varphi\rangle a_{ij} \langle\varphi| \Rightarrow A' = u A u^{+}$$

Eigenvalue equation

$$\hat{A}\hat{\psi} = \lambda \vec{\psi}$$

→ solve for eigenvalues then - eigenvectors, orthogonalize eigenvectors

if A - hermitian, $A u^{-1} u c = \lambda c$

$$(u A u^{-1})(uc) = \lambda (uc)$$

diagonal matrix

same eigenvalue

Also, $\langle \varphi_m | A \varphi_n \rangle = \varphi_m \lambda_n \varphi_n = \lambda_n \langle \varphi_m | \varphi_n \rangle$

$$\langle A^\dagger \varphi_m | \varphi_n \rangle^* = \langle \varphi_n | A | \varphi_m \rangle = \lambda_m^* \langle \varphi_n | \varphi_m \rangle^*$$

$$\underbrace{(\lambda_n - \lambda_m^*)}_{\text{real}} \underbrace{\langle \varphi_m | \varphi_n \rangle}_{\text{mutually orthogonal}} = 0$$

In general, any normal matrix: $[A, A^\dagger] = 0$ can be diagonalized by unitary transformation,

Normal matrices also have orthogonal eigenvectors.