Note Title 2/17/2011

Consider a linear operator:

$$\mathcal{L}u = p_0(x)u'' + p_1(x)u' + p_2(x)u$$

first $p-i$ derivatives of p_i are continuous;

$$p_0(x) \neq 0 \quad \forall\, a<x<b$$

Define quadratic form:

$$\langle u|\mathcal{L}|u\rangle = \int_a^b u(x)\,\mathcal{L}u(x)\,dx =$$

$$= \int_a^b \left(u p_0 u'' + u p_1 u' + u p_2 u\right)dx = u(p_0 u')\Big|_a^b + u p_1 u\Big|_a^b -$$

$$+ \int_a^b \left(-u'(p_0 u)' - u(p_1 u)' + u p_2 u\right)dx = \quad =$$

$$= u p_1 u\Big|_a^b + u p_0 u'\Big|_a^b - u(p_0 u)'\Big|_a^b + \int \left[(p_0 u)'' - (p_1 u)' + p_2 u\right]u\,dx =$$

$$= u(p_1 - p_0')u\Big|_a^b + \int\left[(p_0 u)'' - (p_1 u)' + p_2 u\right]u\,dx =$$

$$= u(p_1 - p_0')u + \int\left[(p_0' u + p_0 u')' - p_1' u - p_1 u' + p_2 u\right]u\,dx =$$

$$p_0'' u + p_0' u' + p_0' u' + p_0 u''$$

$$= u(p_1 - p_0')u + \int\left[p_0 u'' + u'(2p_0' - p_1) + u(p_2 + p_0'' - p_1')\right]u\,dx$$

$$\Rightarrow \quad 2p_0' - p_1 = p_1 \quad \Rightarrow \begin{cases} p_0' = p_1 \\ p_0' = p_1 \end{cases}$$

$$p_0'' - p_1' = 0 \quad \Rightarrow$$

$$= \bar{\mathcal{L}} u = \frac{d^2}{dx^2}(p_0 u) - \frac{d}{dx}(p_1 u) + p_2 u$$ - adjoined op.

Self-adjoined operator $(p_0' = p_1)$:

$$(u|\mathcal{L}u) = (\bar{\mathcal{L}}u|u) \quad \text{or} \quad (v|\mathcal{L}u) = (\mathcal{L}v|u)$$

$$\bar{\mathcal{L}}u = \mathcal{L}u = \frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u(x),$$

where $p(x) = p_0(x)$; $q(x) = p_2(x)$

$$\mathcal{L}u_\lambda + \lambda w(x) u_\lambda = 0$$

eigenfunction

weight-function

eigenvalue

with $\langle v|u\rangle = \int_a^b v^*(x) w(x) u(x)\, dx$

Note: we must select appr. bound. cond.
for self-Adjoined operators:

$$\int v^* \mathcal{L} u\, dx = \int u (\mathcal{L} v)^* dx$$ ← Hermitian op.

$$\equiv (v \mathcal{L}^+ | u\rangle$$

$$\langle \mathcal{L} \rangle = \int \psi^* \mathcal{L} \psi\, dx$$

Orthogonality of eigenfunctions & real eigenvalues:

$$\mathcal{L} u_i = \lambda_i u_i w$$

$$\int u_j^* \mathcal{L} u_i dx = \lambda_i \int u_j^* u_i w dx$$

$$\mathcal{L}^* u_j^* = \lambda_j^* u_j^* w$$

$$\int u_i \mathcal{L}^* u_j^* = \lambda_j^* \int u_j^* u_i w dx$$

$$0 = \underbrace{(\lambda_i - \lambda_j^*)}_{\lambda_i = \lambda_i^* \Rightarrow \lambda_i \in \mathbb{R}} \underbrace{\int u_j^* u_i w dx}_{\delta_{ij} \cdot \int |u_i|^2 dx}$$

Note: any lin. comb. of eigenfuncs is an eigenfun

Assume that we have N linearly independent eigenfunctions u_i

To construct a set of orthonormal funcs:

$$\varphi_1(x) = \frac{u_1(x)}{\sqrt{\int |u_1|^2 w dx}}$$

$$\psi_2 = u_2 + a_{21} \varphi_1(x) \Big\} \varphi_1^* = 0$$

$$\Rightarrow a_{21} = \int \varphi_1^* u_2 w dx$$

$$\varphi_2 = -\frac{\psi_2(x)}{\left[\int w|\psi_2| dx\right]^{1/2}}$$

$$\psi_3 = u_3 + a_{31}\varphi_1 + a_{32}\varphi_2 ; \quad \varphi_3 = \frac{\psi_3}{\int |\psi_3|^2 w\, dx}$$

...

$$a_{ni} = -\int \varphi_i^* u_n w\, dx$$

In particular,

eigenvalue of Helmholtz eq:

$$\nabla^2 u_j = k_j^2 u_j$$

Green's function eq:

$$\nabla^2 G + k^2 G = -\delta(r_1 - r_2)$$

$$G = \sum_i a_i(r_2) u_i(r_1)$$

$$\Rightarrow \sum(-k_i^2 + k^2) a_i u_i = -\delta(r_1 - r_2)$$

$$\Rightarrow a_j (k^2 - k_j^2) = -u_j(r_2)$$

$$\Rightarrow G = \sum \frac{u_i(r) u_i(r_2)}{k_i^2 - k^2}$$

Green's functions:

(1) variation of constants

Consider the inhomogeneous linear D.E:

$$\mathcal{L} y = f(\bar{r})$$

B.C.

The GF is a solution of:

$G(r,r_i) \underset{r \to \infty, r}{\to} 0$

$$\mathcal{L}\, G(r_1, r_2) = \delta(\bar{r}_1 - \bar{r}_2)$$

$$\int dr_2 \mathcal{L}\, G(r_1, r_2)\, f(r_2) = \int \delta(r_1 - r_2)\, f(r_2)\, dr_2$$

$$\mathcal{L} \int dr_2\, G(r_1, r_2) f(r_2) = f(r_1)$$

$$\Downarrow$$

$$y(r) = \int dr_2\, G(r_1, r_2) f(r_2)$$

Example:

$$\nabla^2 \varphi = -4\pi\rho$$

$$\nabla^2 \left(\frac{1}{|r - r_2|} \right) = -4\pi\, \delta(r - r_2)$$

$$\varphi = \int \frac{\rho(r_2)\, dr_2}{|r - r_2|}$$

General approach 1D problems

$$\mathcal{L} = \frac{d}{dx}\left(p \frac{d}{dx} G\right) + q G = \delta(t - t_0)$$

Solution: $\mathcal{L} G_1 = 0, \quad t < t_0$

$\mathcal{L} G_2 = 0, \quad t > t_0$

$$-p\left(\frac{dG_1}{dx} - \frac{dG_2}{dx}\right) = 1 \Rightarrow \frac{dG_2}{dx} - \frac{dG_1}{dx} = \frac{1}{p(t)}$$

Relationship between diff. eqs & integral eqs

There is a correspondence between a ODE (with B.C) & integral equation:

$$y'' + A(x) y' + B(x) y = g(x),$$

$$y(a) = y_0 \, ; \; y'(a) = y_0'$$

$$\int_a^x dt \, y'' = g(x) - A(x) y' - B(x) y$$

$$\int_a^x du \; y'(x) - y_0' = \int_a^x g(t)\, dt - \int_a^x \underbrace{A}_{u} \underbrace{y' dt}_{dv} - \int_a^x B y \, dt =$$

$$= \int_a^x (g - By)\, dt - A y \Big|_a^x + \int_a^x A' y \, dt$$

$$y(x) - y_0 - y_0'(x-a) = -\int_0^x du\,(A\,y) + A_0 y_0 (x-a) + \int_0^x du \int_0^u \left(g + (A' - B)y\right) dt$$

Now, $\displaystyle\int_0^x du \int_0^u f(t)\,dt = \int_0^x (x-t)\,f(t)\,dt$.

$$\frac{d}{dx} \Rightarrow \int_0^x f(t)\,dt \qquad \frac{d}{dx}\left[x \int_0^x f(t)\,dt - \int_0^x t f(t)\,dt\right] = $$

$$= \int_0^x f(t)\,dt + \cancel{x f(x)} - \cancel{x f(x)}$$

$\Rightarrow$

$$y(x) = \underbrace{y_0 + (y_0' + A_0 y_0)(x-a) + \int_0^x (x-t)\,g(t)\,dt}_{f(x)} + \int_0^x dt\, y(t) \cdot \underbrace{\left[A(t) + (x-t)(A' - B)\right]}_{k(x,t)}$$

When B. conditions specify $y(a)$ & $y(b)$, the latter equation allows to relate y_0' to $y(b)$, transforming the original eq to

$$y(x) = f(x) + \int_a^b dt\, k(x,t)\, y(t)$$